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VISAGG
Visualization and Graphics Group

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Compactly Supported Biorthogonal Wavelet Bases on the BCC Lattice

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Introduction

- Motivation
  - What non-Cartesian computing is, and why should you care about it
- Background
  - Box splines and biorthogonal wavelet filter banks
- Methodology
  - Let’s make a wavelet
- Results
  - Let’s compress data

Motivation

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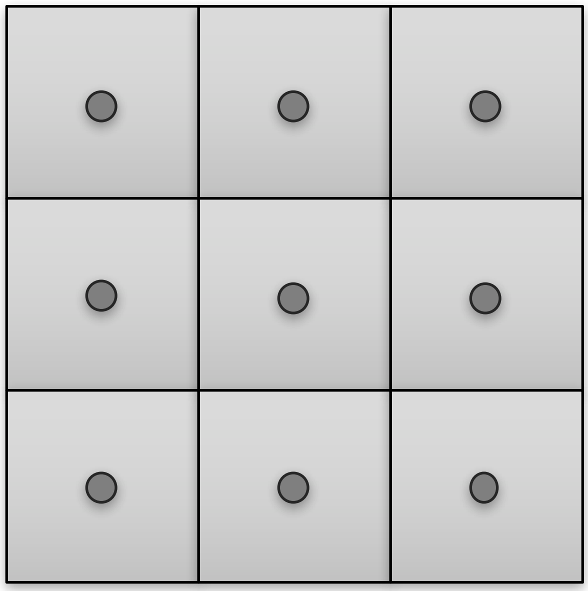
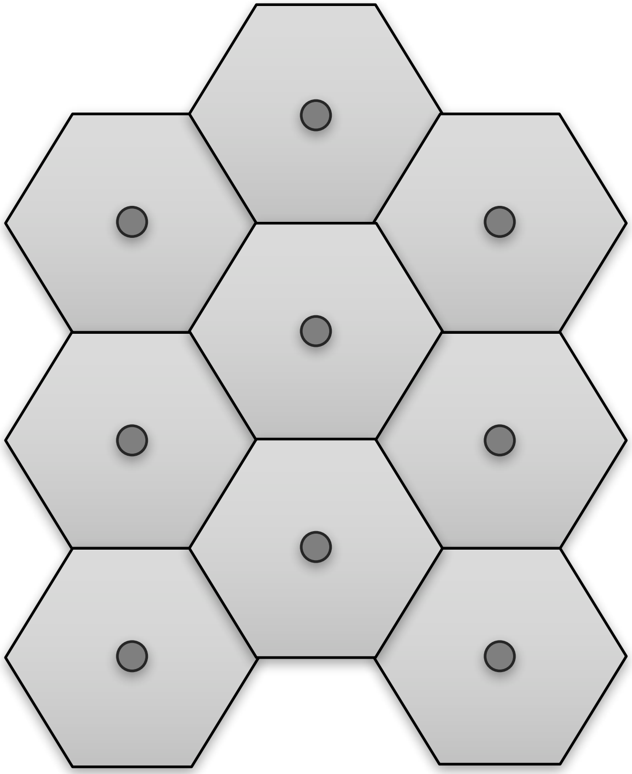


Planar Cartesian Lattice

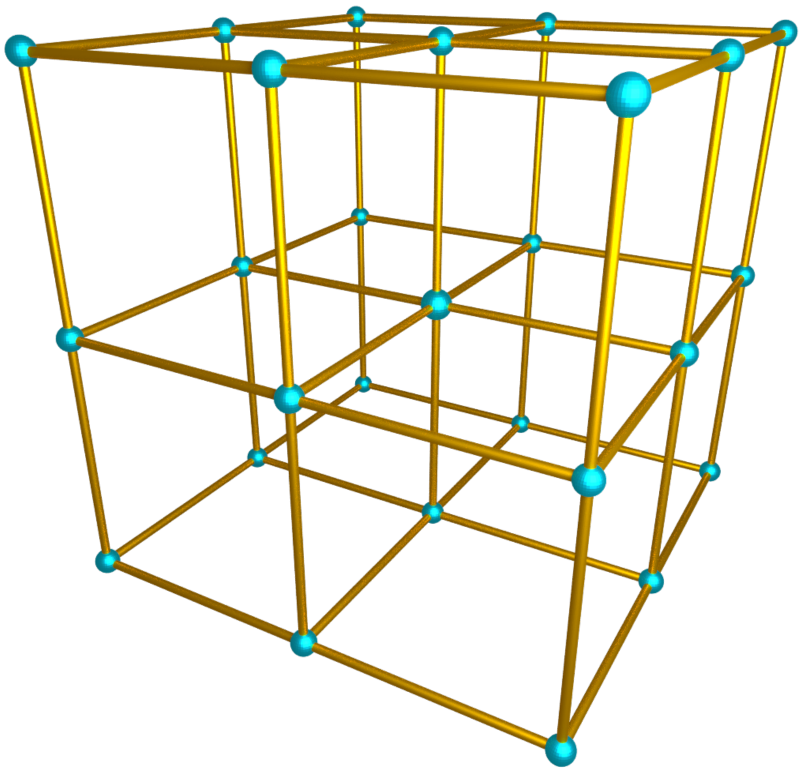


Hexagonal Lattice

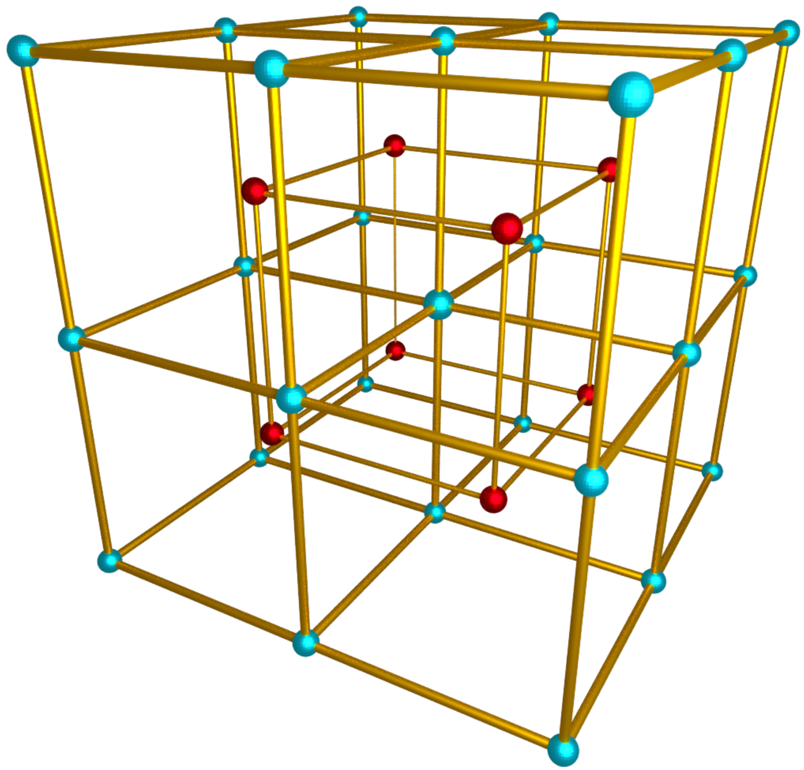
Motivation

- Why bother with non-Cartesian schemes?
  - Indexing of data is ambiguous
  - Interpolation is more difficult
  - Data processing is more difficult
- Benefits
  - Better approximation properties
  - Higher level of isotropy
  - Fewer samples needed for reconstructions

Motivation



Cubic Cartesian Lattice



Body Centered Cubic (BCC) Lattice

- What has been done?
  - Interpolation [Entezari *et al*. 2008, Kim *et al*. 2013, Csebfalvi *et al.* 2013]
  - Data reconstruction [Xu *et al*. 2012]
  - Gradient approximation [Alim *et al.* 2010, Hossain *et al*. 2011]
  - Fast Fourier Transforms [Alim *et al*. 2009]
- Wavelets are still missing
  - Compression
  - De-noising
  - Progressive Rendering

- Background
  - Box splines: natural interpolants on non-Cartesian grids
  - Biorthogonal wavelet filter banks: decompose high and low frequencies

- Box Splines
  - Multivariate extension to B-splines
  - Compact, Smooth
  - Piecewise Polynomial

To define a box spline, start with a direction matrix

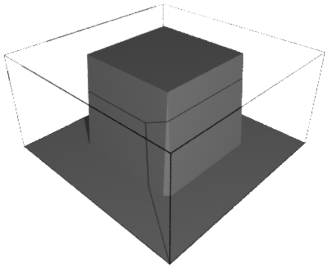
$$\Xi = \begin{bmatrix} | & | & & | \\ \vec{\xi}_1 & \vec{\xi}_2 & \cdots & \vec{\xi}_n \\ | & | & & | \end{bmatrix}$$

$\vec{\xi}_i$ is a two or three dimensional column vector in our case

- Box Splines

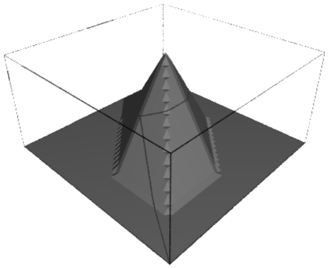
Convolutional definition, similar to B-splines

$$M_{\Xi}(x) = \int_0^1 M_{\Xi\backslash\xi}(x - t\xi)dt \qquad \text{with} \qquad M_{[\,]}(x) = \delta(x)$$



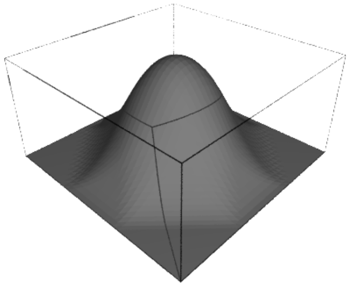
$$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

Box function



$$\begin{bmatrix}1 & 0 & 1\\0 & 1 & 1\end{bmatrix}$$

Courant element

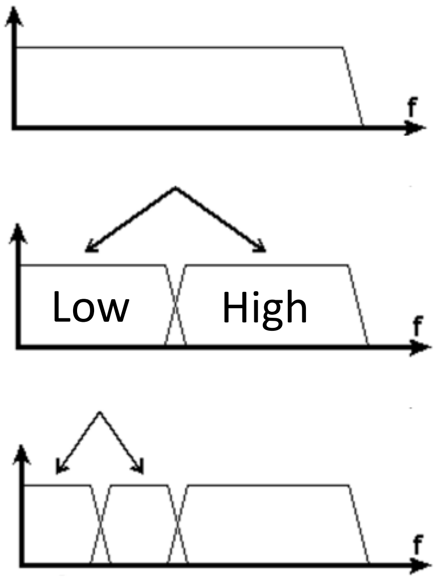


$$\begin{bmatrix}1 & 0 & 1 & -1\\0 & 1 & 1 & 1\end{bmatrix}$$

Zwart Powell element

Some box splines are very natural interpolants for certain lattices

- Biorthogonal Wavelet Filter Banks
  - A motivating example, in one dimension

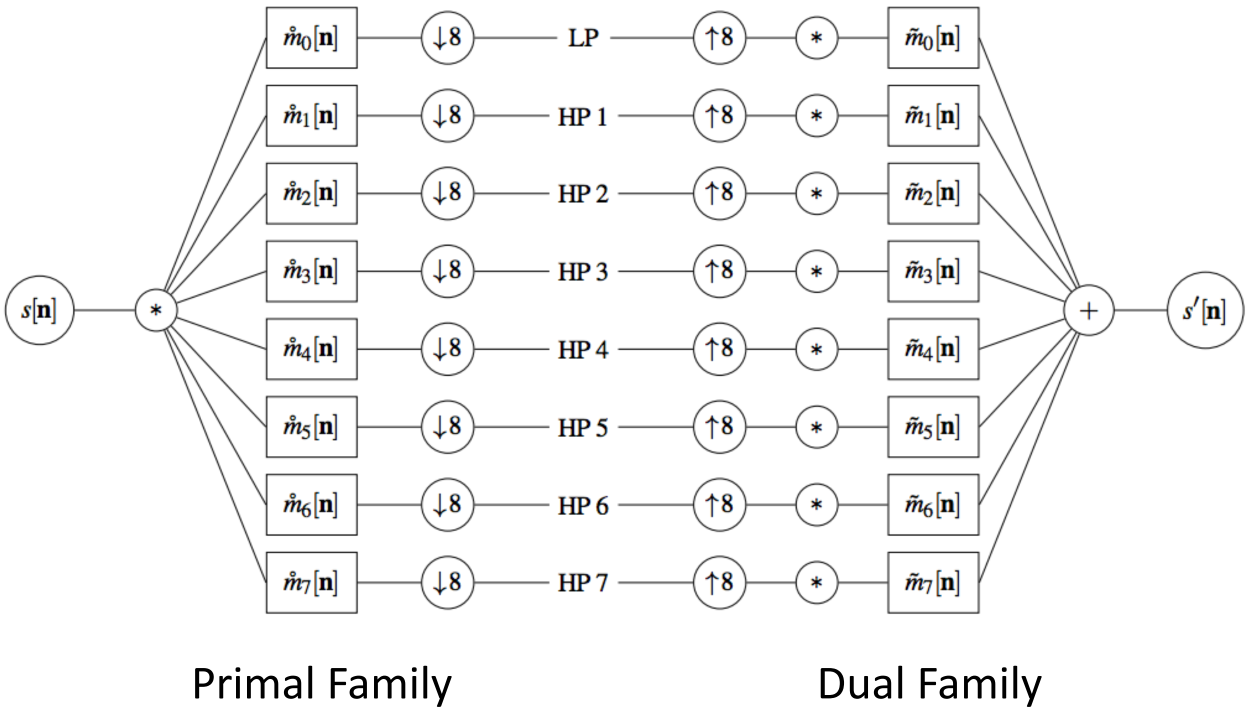


Original function (in Fourier domain)

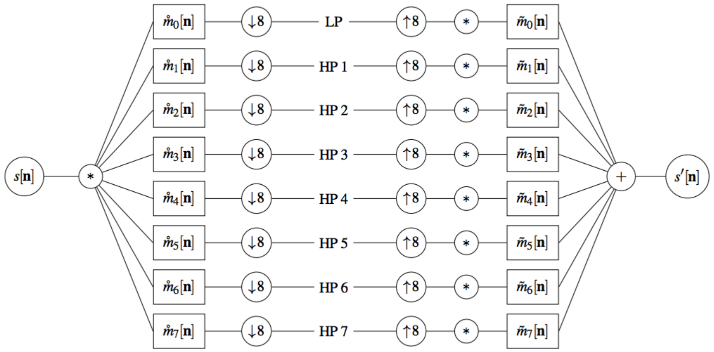
After one level of decomposition

After two levels of decomposition

- Wavelet Filter Banks
  - In 3D, for a uniform scaling factor of two, we need 8 channels, 1 low pass, 7 high pass



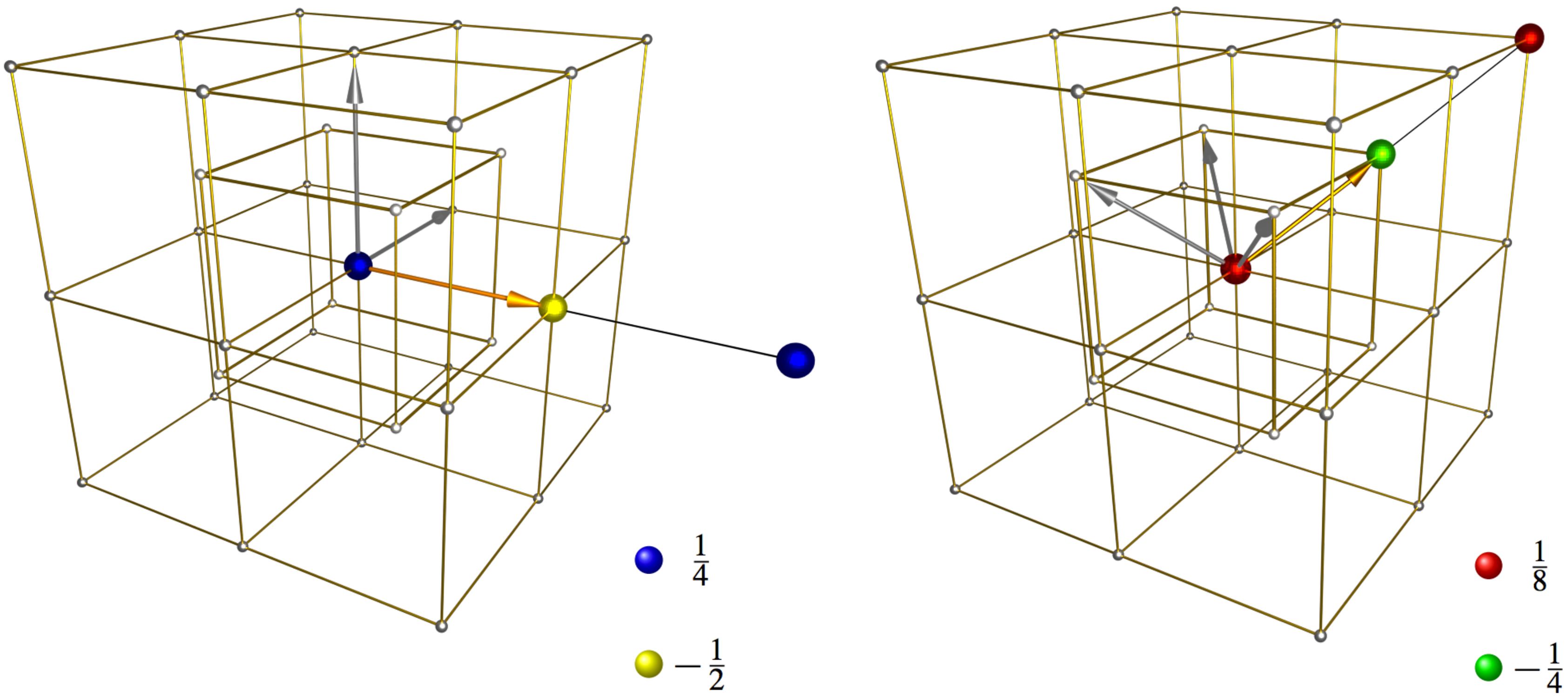
- Biorthogonal Wavelet Filter Banks
  - Decompose a signal in terms of high and low frequency content (enforce perfect reconstruction)
  - Preferably compact filters
  - Number of wavelets depends on number of channels



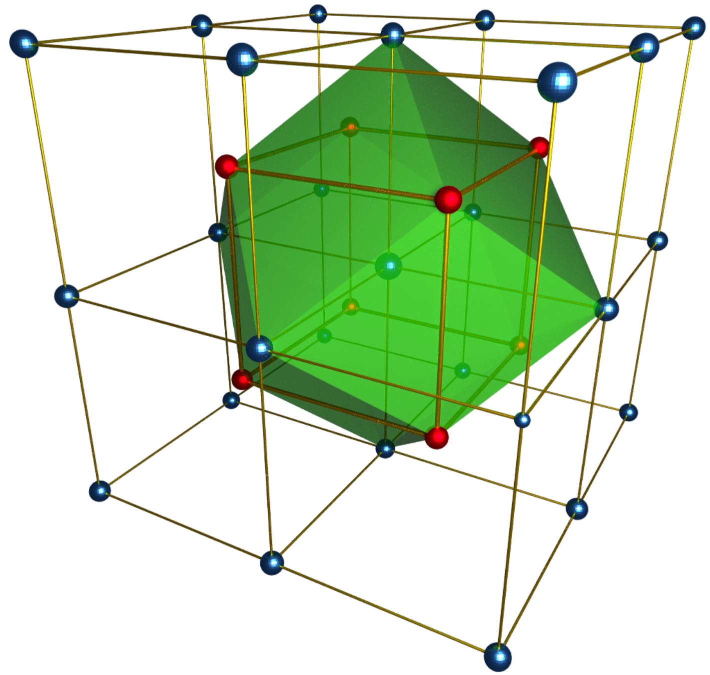
$$\sum_j M_j(\mathbf{z}^{-1} e^{-\iota\pi_i}) \tilde{M}_j(\mathbf{z}) = \delta[i] \quad \text{for} \quad i \in \{0,\ldots,7\}$$

- Our Methodology
  - There are a lot of degrees of freedom
  - We can exploit some of the symmetry of the BCC lattice
  - We force 7 filters to be geometrically similar, then complete the system of filters, similar to [Cohen et al. 1993]

- Wavelet Filter Banks on the BCC Lattice

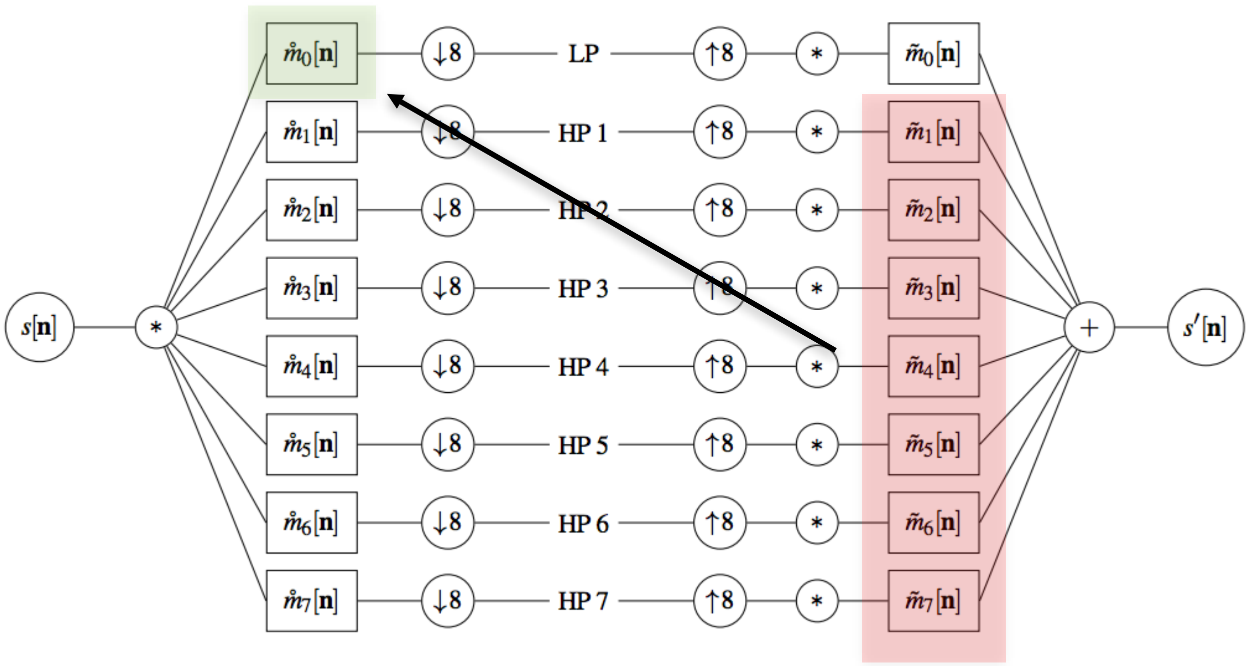


- Wavelet Filter Banks on the BCC Lattice
  - Proposition 3.1 tells us that we can obtain the primal low pass, from these dual high pass filters



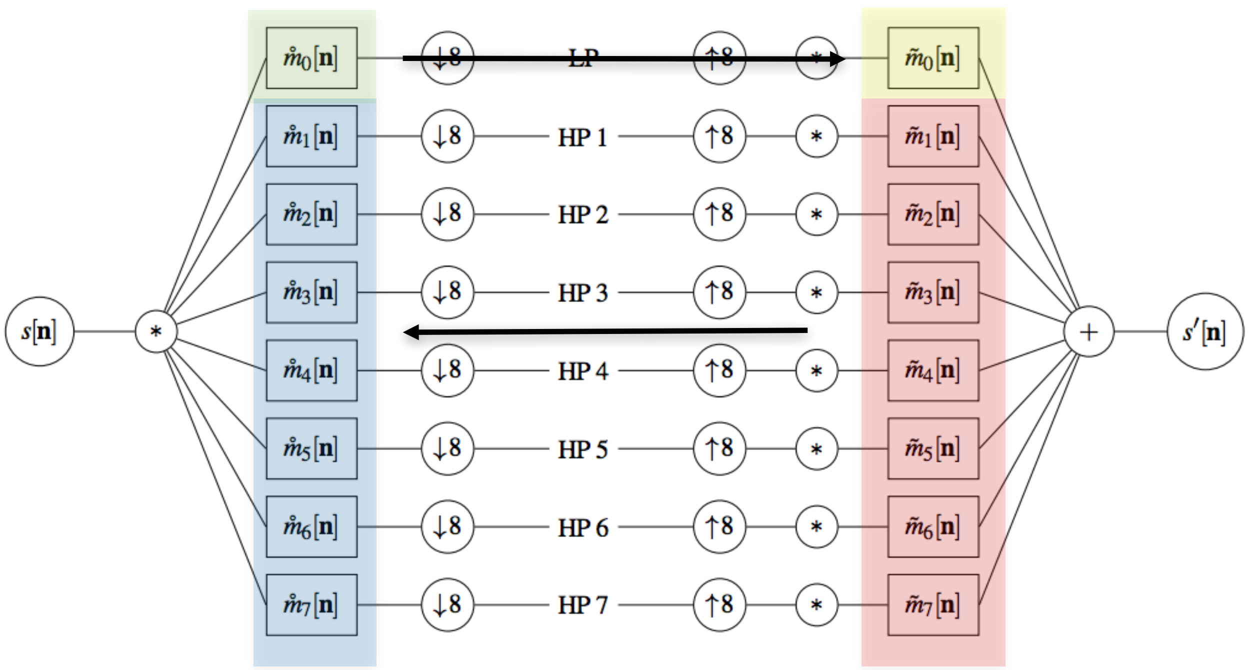
It's a box spline!

- Wavelet Filter Banks

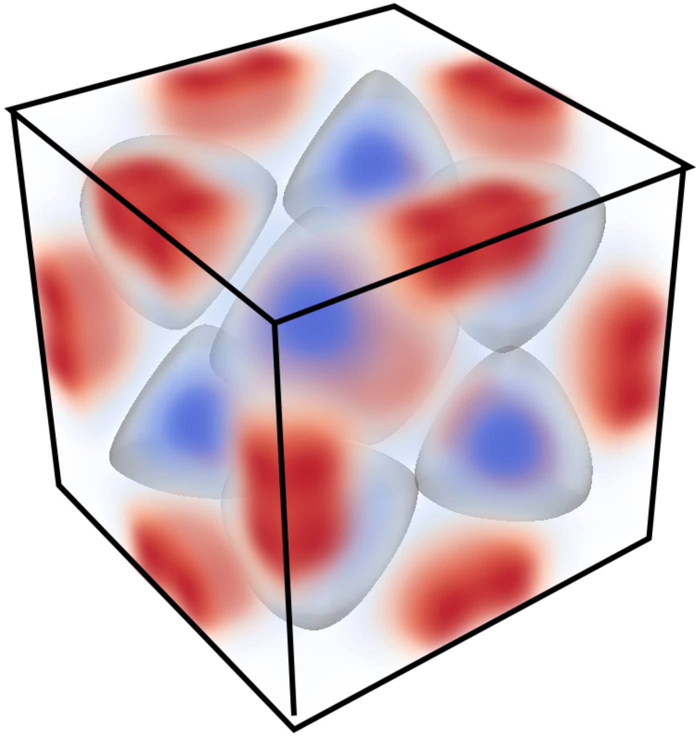


- Wavelet Filter Banks on the BCC Lattice
  - Once we have this primal filter, we need a dual filter
  - Proposition 3.1 gives the condition that such a filter must satisfy
  - We then solve $\boldsymbol{A}(\mathbf{z})\boldsymbol{b}(\mathbf{z}) = (1,0,0,0,0,0,0,0)^T$, which gives the entire filter family
  - To get higher order filters, we can also use the boot-strapping procedure of [Cohen *et al.* 1993]

- Wavelet Filter Banks

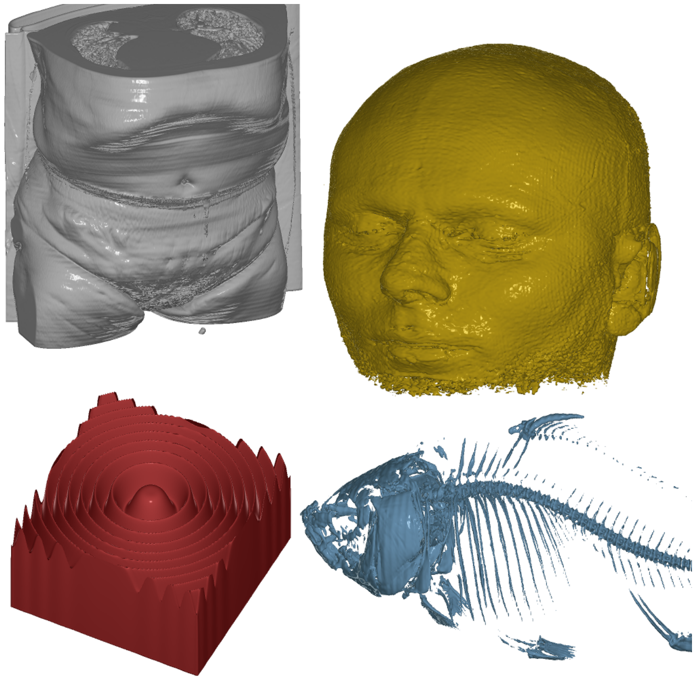


- Higher order filters
  - Some applications need higher order filters (filters that are more smooth in the limit)
  - Boot-strapping (juggle filters and convolutions)

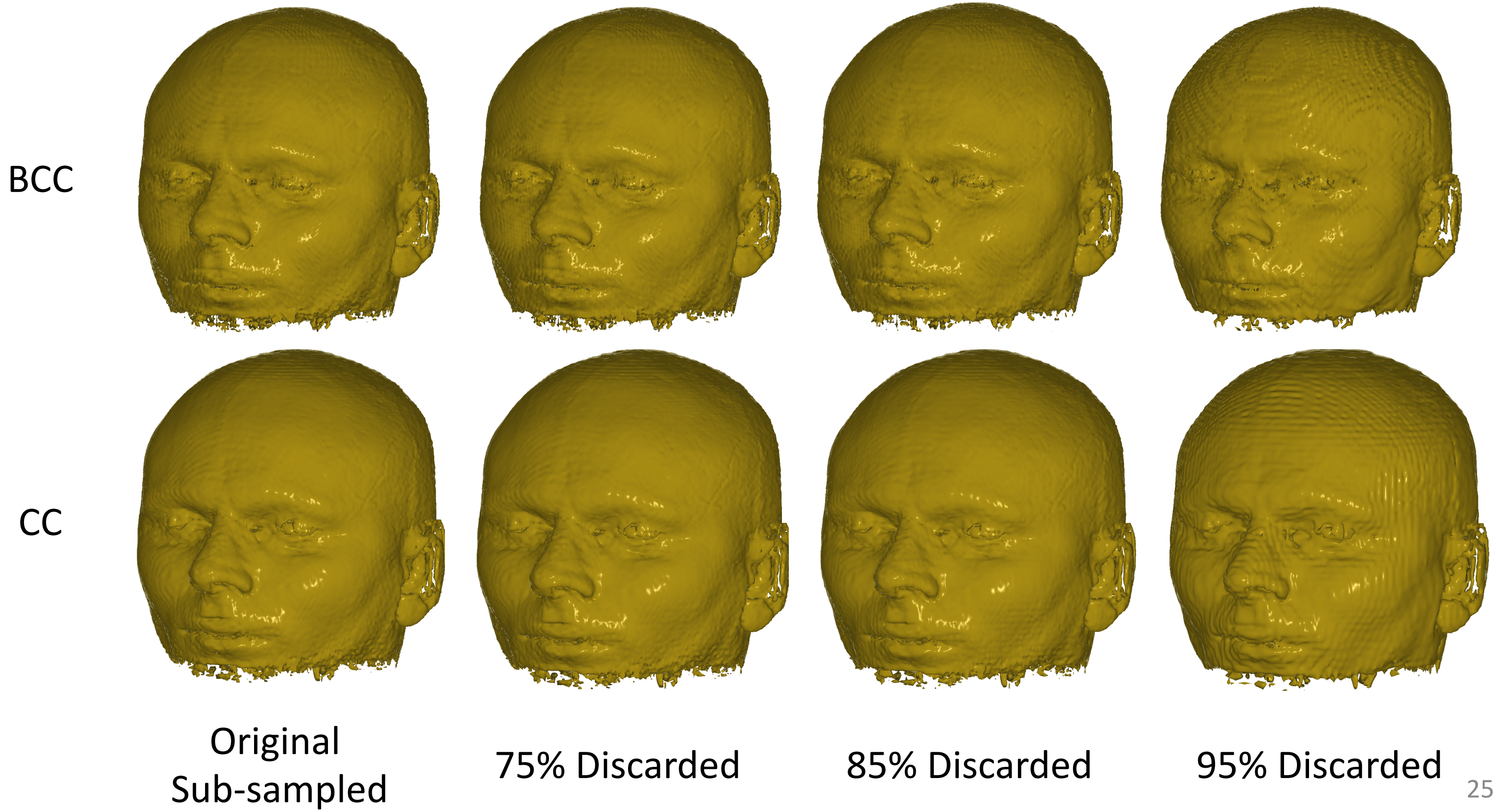


- Compression via Thresholding
  - Perform hierarchical wavelet decomposition
  - Discard coefficient such that only a percentage of the largest coefficients remain
  - Reconstruct the volumetric image

- Datasets
  - Start with high resolution data sets, resample them onto a lower resolution Cartesian and BCC grid
  - Sampling density is roughly the same
  - Marschner Lobb is known in closed form
  - Measure peak signal to noise ratio (PSNR)

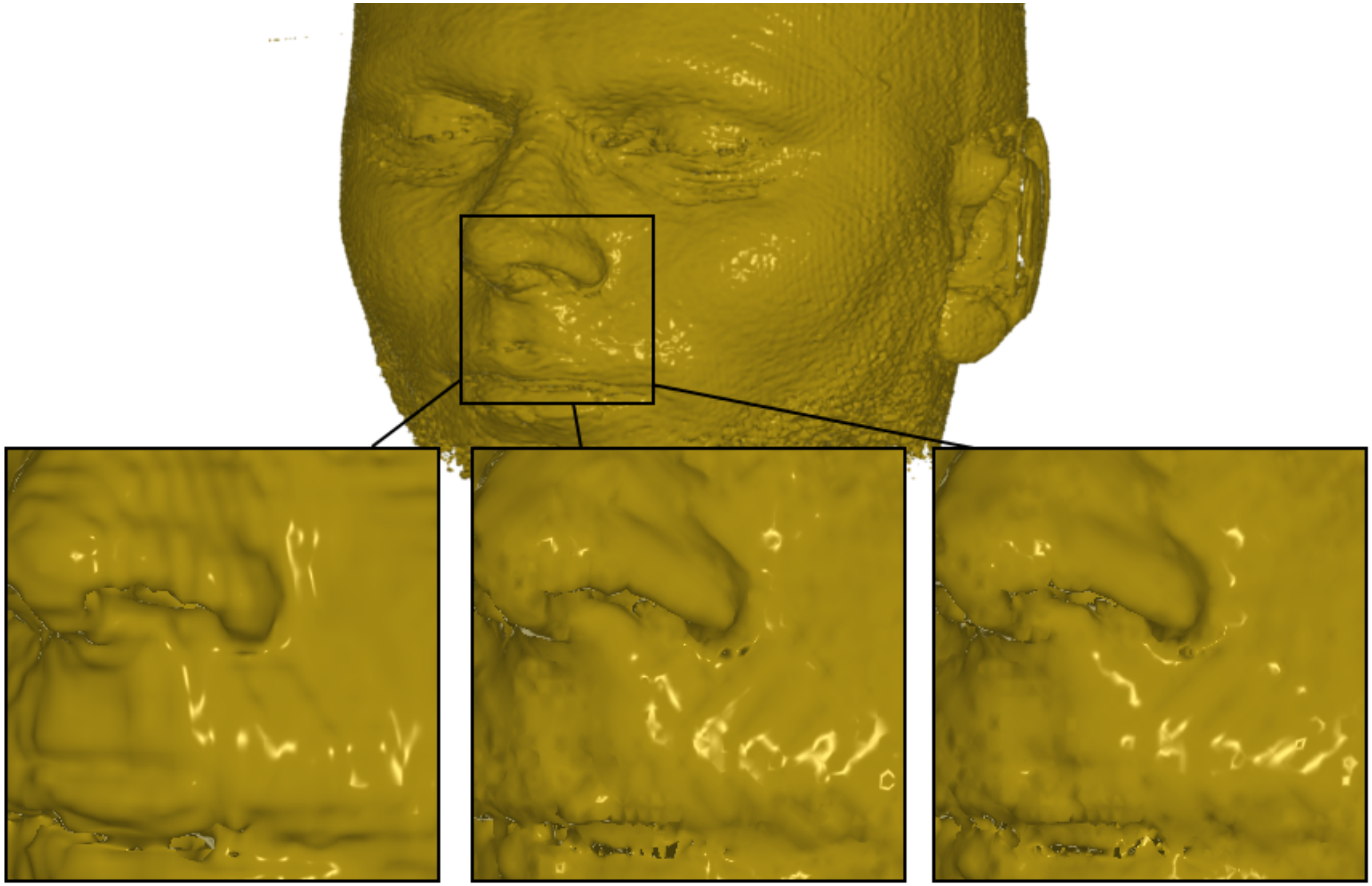


- Qualitative Results
  - Head Dataset



Experiments

- Qualitative Results (95% Removed)

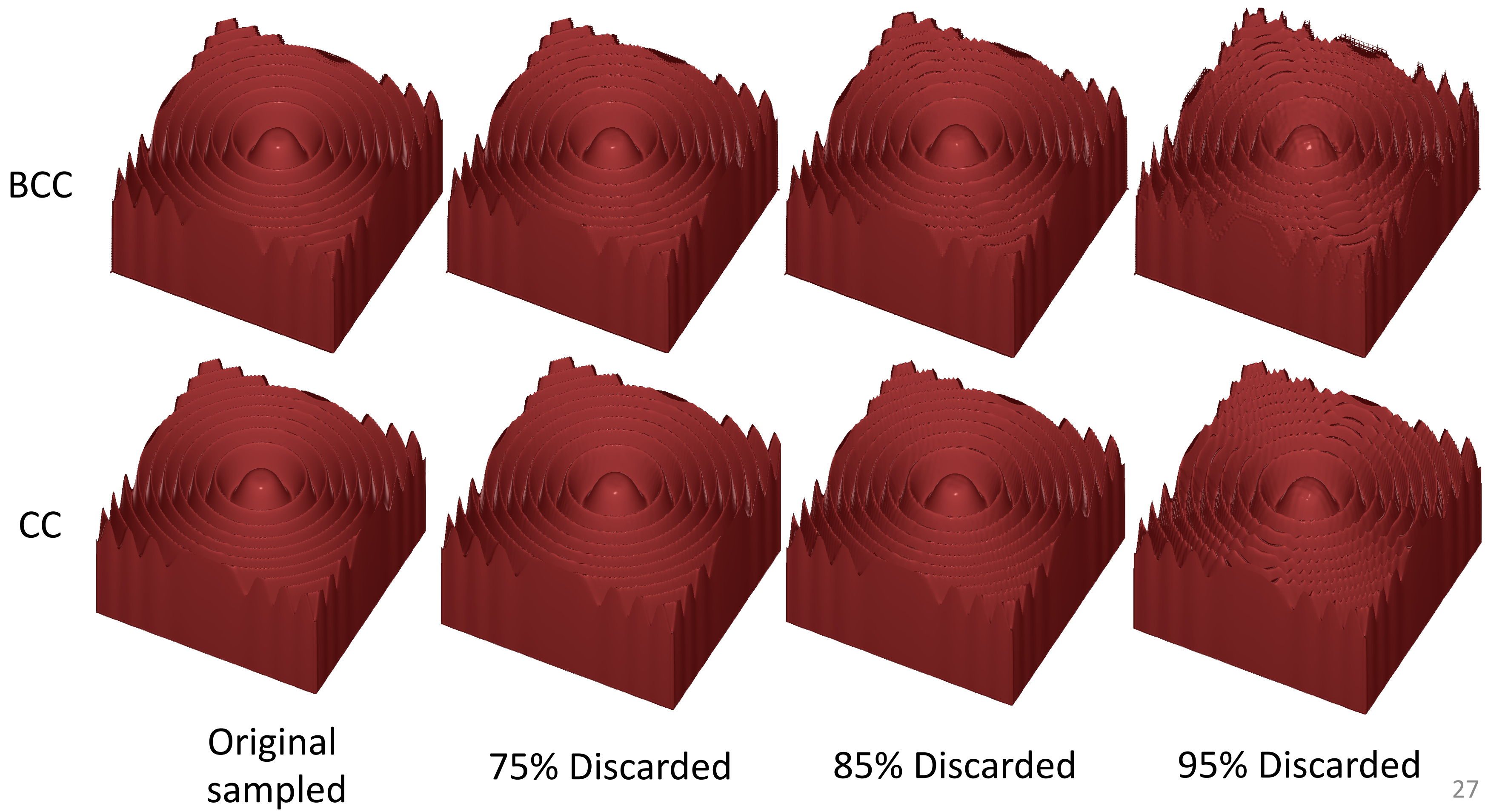


Trilinear Cartesian

Linear BCC

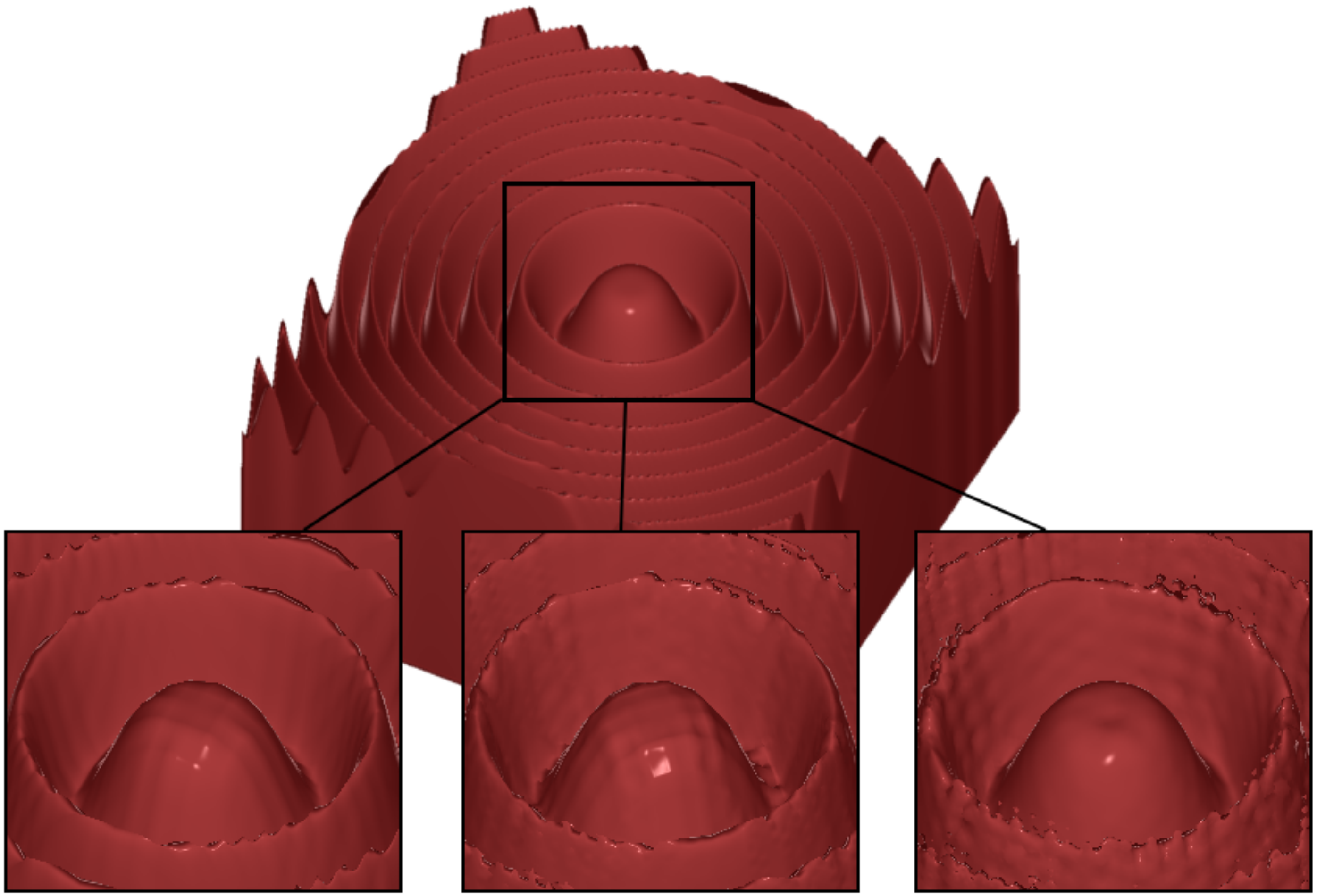
Higher order BCC

- Qualitative Results
  - Marschner Lobb function



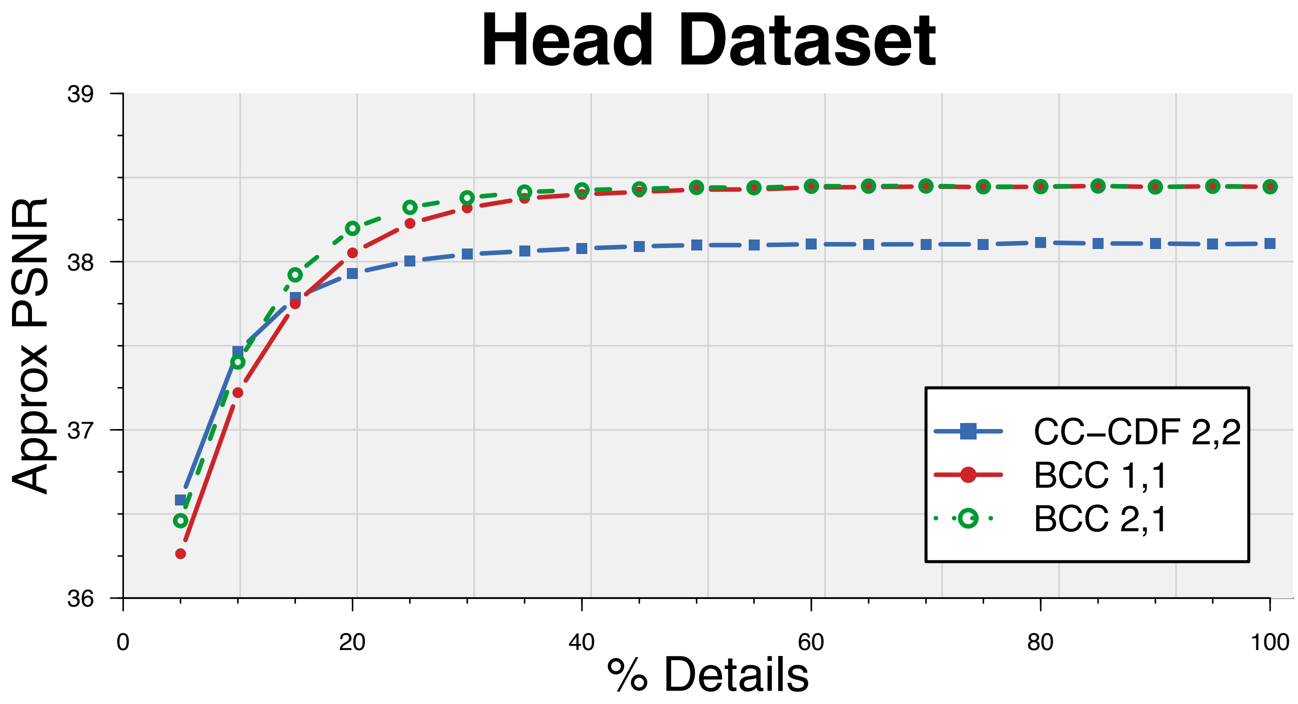
Experiments

- Qualitative Results (95% Removed)

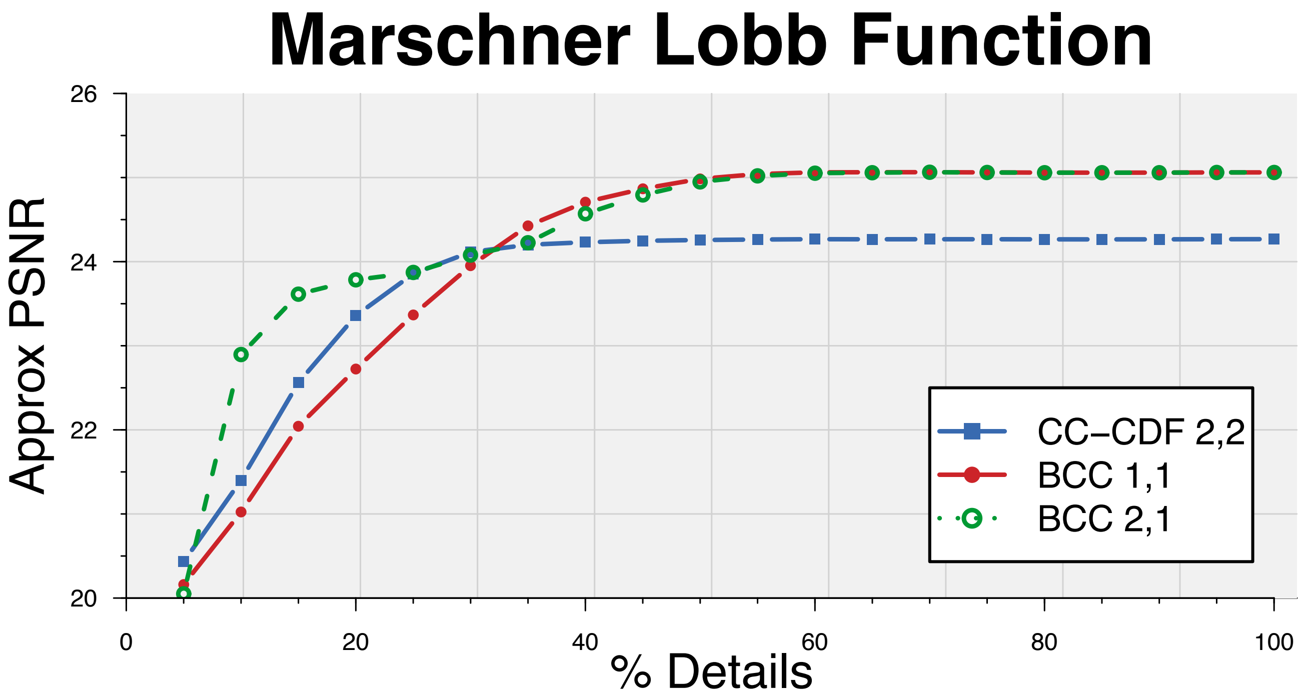


Trilinear Cartesian | Linear BCC | Higher order BCC

- Quantitative Results



- Quantitative Results



- Contributions
  - We derived a family of geometrically appropriate wavelet filter banks on the BCC lattice
  - Our wavelet filter banks perform at least on par with Cartesian style CDF wavelets

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Acknowledgements

- References

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