Gradient Estimation Revitalized

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SFU SIMON FRASER UNIVERSITY
THINKING OF THE WORLD

gruvi graphics + usability + visualization

GREYC

Motivation

- Good renderings need good gradients

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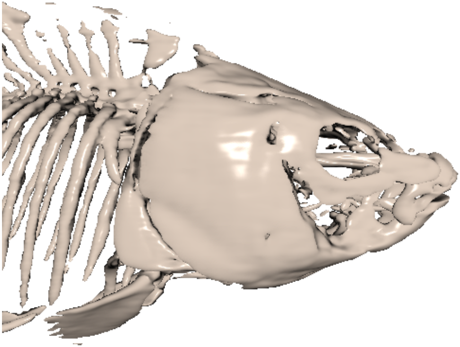

Finite Differencing

Motivation

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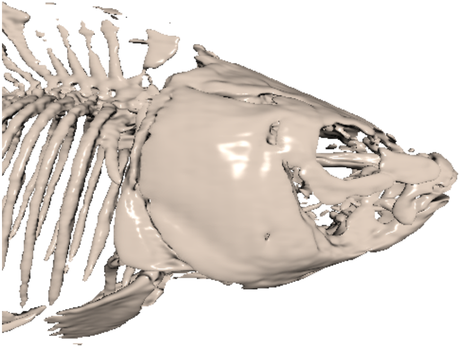
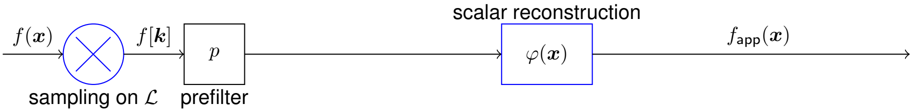


Orthogonal Projection

Related Work: Quantitative Analysis

- Quantitative Fourier analysis of scalar reconstruction schemes [Unser and Blu '99]
- Linear interpolation *revitalized* [Blu *et al.* '04]
- Extension to derivatives in 1D [Condat and Möller '09]

Overview ($\mathbb{R}^3$)



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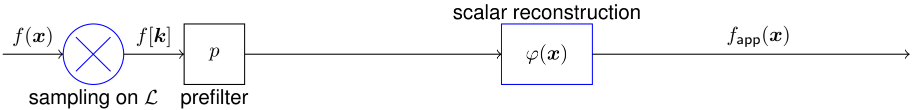
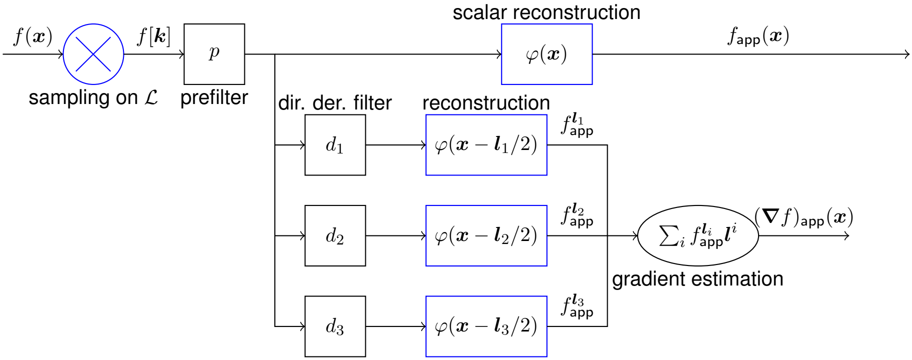


Why prefilter?

1. Ensures approximation and original function agree at the lattice sites
2. Exploits the full approximation power of φ

Overview ($\mathbb{R}^3$)



$\boldsymbol{l}_i$: Principal lattice directions

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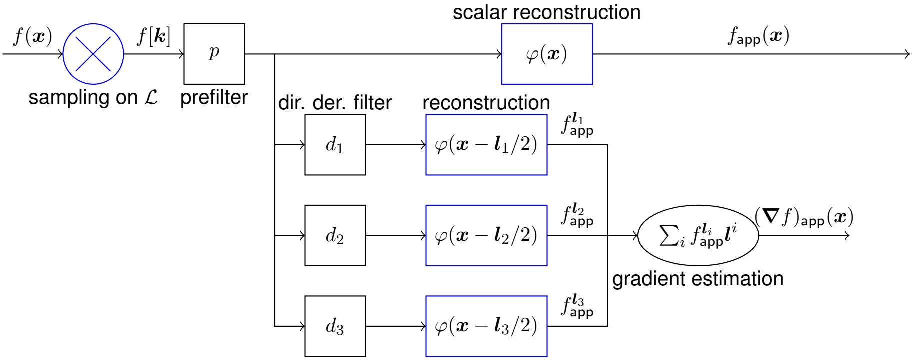
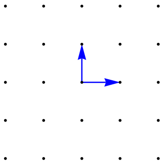


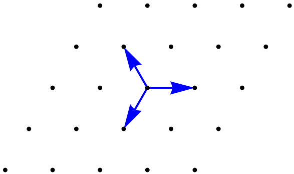
$\boldsymbol{l}_i$: Principal lattice directions

- More general case considered in the paper

Principal Directions

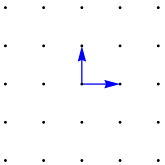


2D Cartesian: 2



Hexagonal: 3

Principal Directions



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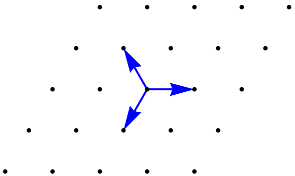

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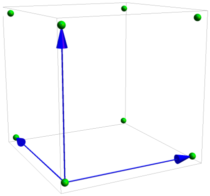
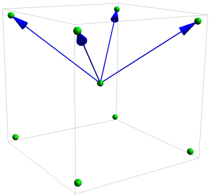


CC: 3



BCC: 4

Formal Description

- Approximate derivatives in the principal directions $\boldsymbol{l}_i$
- Interested in a digital filter that approximates in the shift-invariant space $\mathbb{V}(\mathcal{L}_h, \varphi^i)$, i.e.

$$\partial_{\boldsymbol{l}_i} f(\boldsymbol{x}) \approx f_{\text{app}}^{\boldsymbol{l}_i}(\boldsymbol{x}) = \sum_{\boldsymbol{k} \in \mathbb{Z}^s} \frac{1}{h} (f * p * d_i)[\boldsymbol{k}] \varphi_{h,\boldsymbol{k}}^i(\boldsymbol{x})$$

$$\varphi^i(\boldsymbol{x}) := \varphi(\boldsymbol{x} - \frac{\boldsymbol{l}_i}{2}) \text{ and } \varphi_{h,\boldsymbol{k}}^i(\boldsymbol{x}) := \varphi^i(\frac{\boldsymbol{x}}{h} - \boldsymbol{L}\boldsymbol{k})$$

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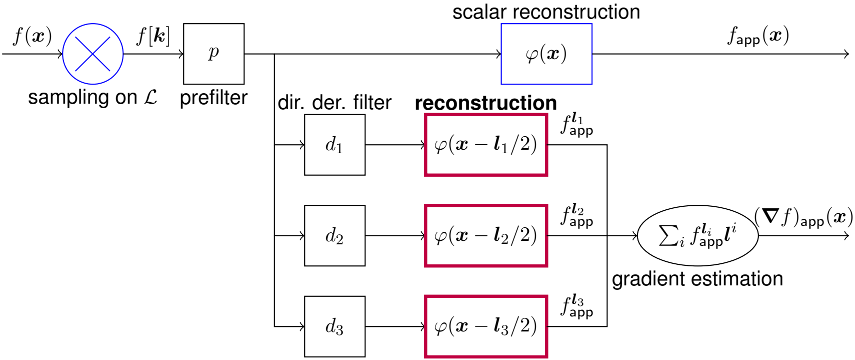
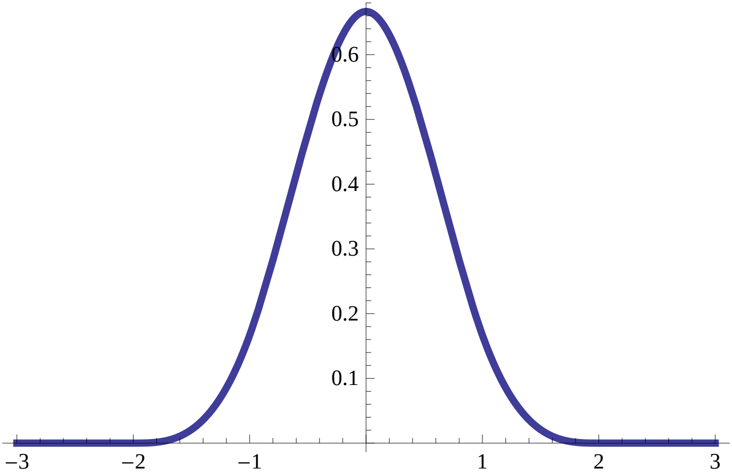
Revitalization



Shift Example (Cubic B-spline)



$\varphi(x)$

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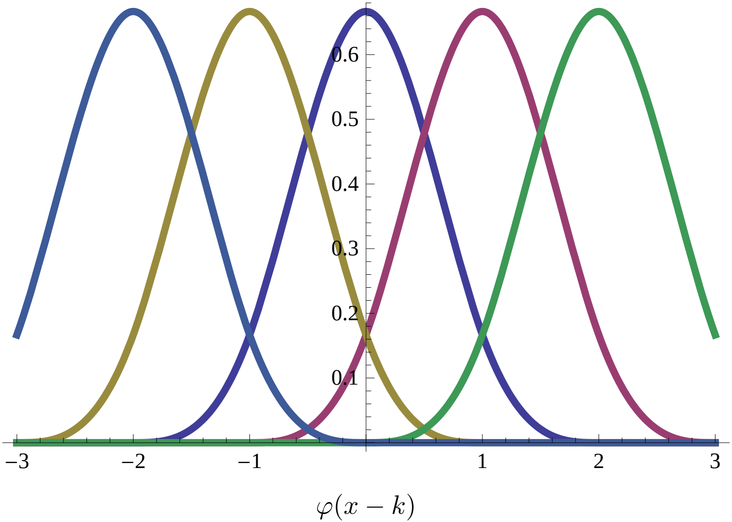

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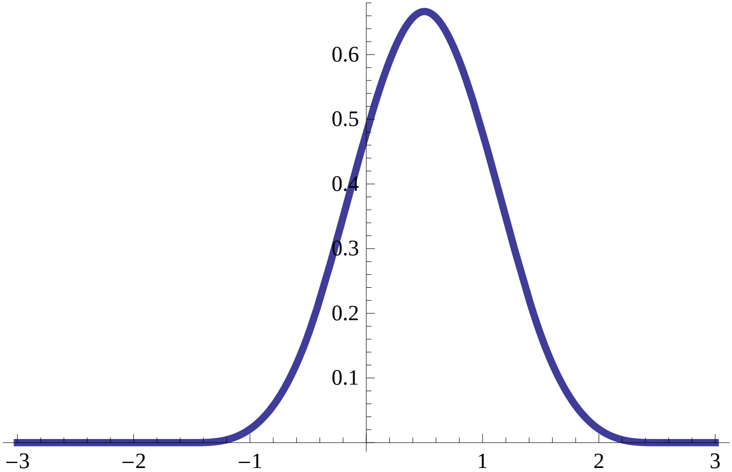

$\varphi^1(x)$

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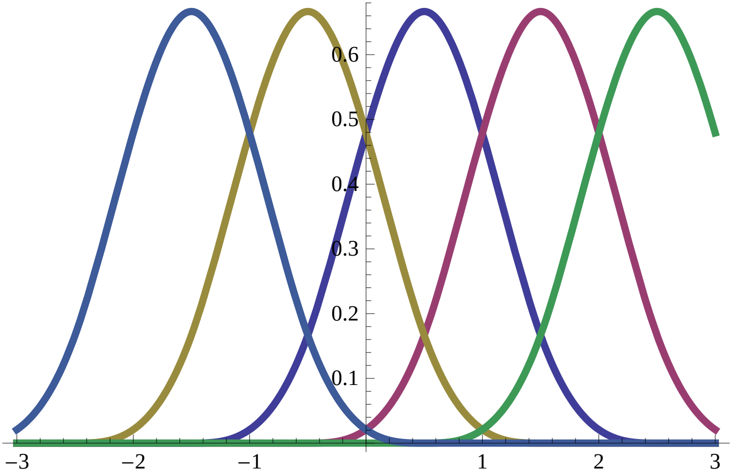
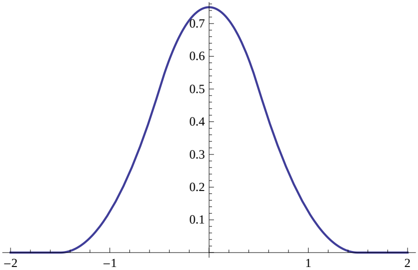
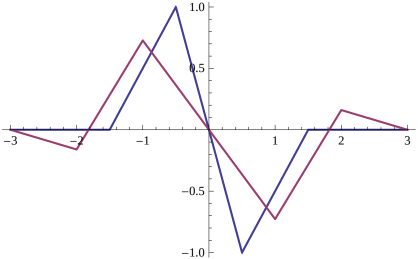


$$\varphi^1(x-k)$$

Why shift? (1D)



$\beta_2(x)$ - Quadratic B-spline



- $\beta_2'(x) = \beta_1(x + \frac{1}{2}) - \beta_1(x - \frac{1}{2})$ (blue)
- $\mathbb{V}(\mathbb{Z}, \beta_1(x))$ can't recover the exact derivative (purple)
- $\mathbb{V}(\mathbb{Z}, \beta_1(x - 1/2))$ can! (blue)

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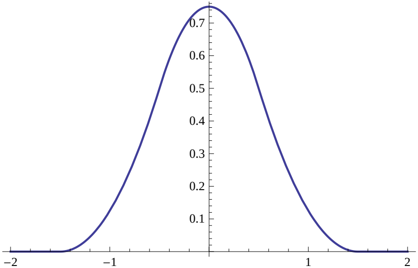

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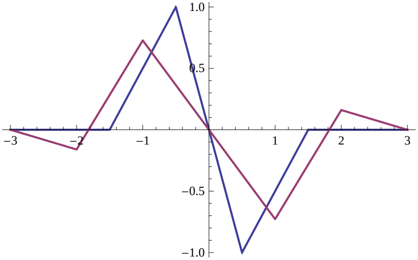

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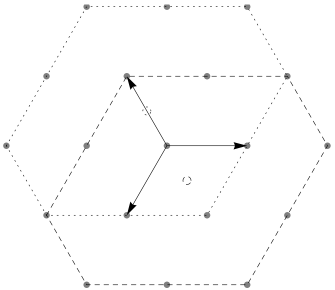
How to shift in higher dimensions?



- Directional derivative of a 4th order hexagonal box spline is a linear combination of two lower order shifted box splines
- Shifts are in the direction of the derivative

1. *Choose s linearly independent principal directions*
2. *Shift the symmetric box spline along those principal directions*

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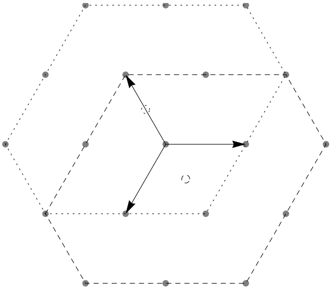

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Error Analysis

How to predict the directional derivative error, given an approximation space $\mathbb{V}(\mathcal{L}, \psi)$ and a filter r

Error Kernel

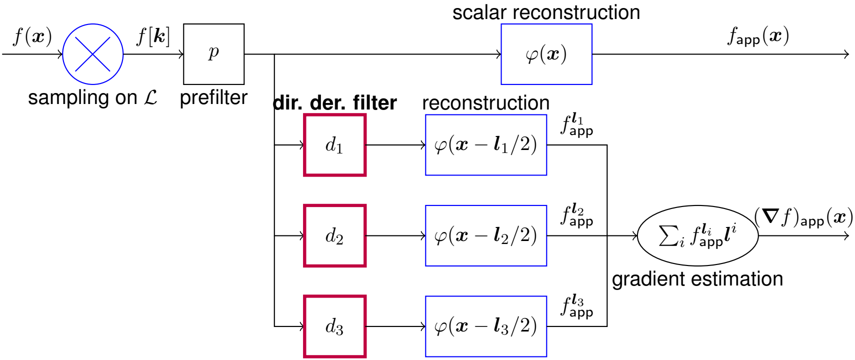
$$E^{\boldsymbol{l}}(\boldsymbol{\omega}) := \underbrace{1 - \frac{\left|\widehat{\psi}(\boldsymbol{\omega})\right|^2}{\widehat{A}_\psi(\boldsymbol{\omega})}}_{E_{\text{min}}(\boldsymbol{\omega})} + \underbrace{\widehat{A}_\psi(\boldsymbol{\omega}) \left| \frac{\widehat{R}(\boldsymbol{\omega})}{j\boldsymbol{l}^{\mathsf{T}}\boldsymbol{\omega}} - \widehat{\overset{\circ}{\psi}}^{\star}(\boldsymbol{\omega}) \right|^2}_{E^{\boldsymbol{l}}_{\text{res}}(\boldsymbol{\omega})}$$

$\boldsymbol{l}$: A principal lattice direction

$\widehat{R} \leftrightarrow r$: Filter applied to samples

$\widehat{A}_\psi$: Autocorrelation sequence

Filter Design



Combined directional derivative filter

$$r_i = (p * d_i)$$

Asymptotic Optimality

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- For a minimum error approximation:
  - $E^{\boldsymbol{l}}_{\text{res}}(\boldsymbol{\omega}) = 0$, not realizable!
- Choose r so that $E_{\text{min}}(\boldsymbol{\omega}) \sim E^{\boldsymbol{l}}_{\text{res}}(\boldsymbol{\omega})$ (as $h \to 0$)
- Plug-in our basis function φ^i and combined filter $r_i = p * d_i$

Optimality Criterion

$$d_i \leftrightarrow \widehat{D_i} = j\boldsymbol{l}_i{}^{\mathsf{T}}\boldsymbol{\omega} \exp(\tfrac{j}{2}\boldsymbol{l}_i{}^{\mathsf{T}}\boldsymbol{\omega}) + O(|\boldsymbol{\omega}|^{n+1})$$

No dependence on φ

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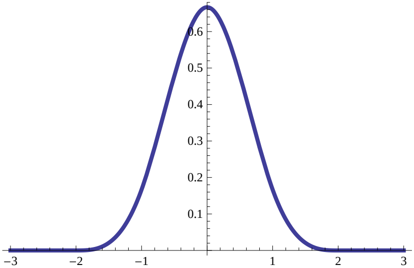
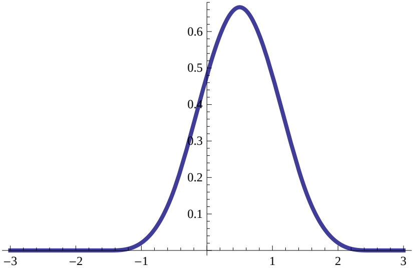
VisIO

Fourth-order 1D Filters d_i



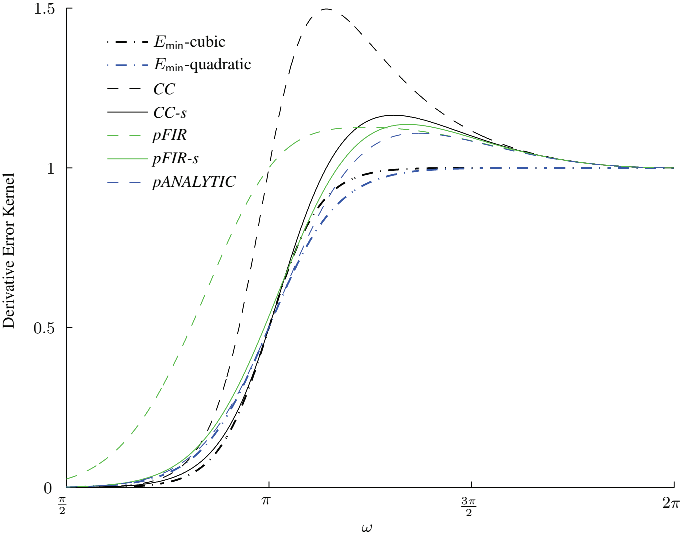
Centered (*pFIR*): $[-\frac{1}{12}, \frac{2}{3}, 0, -\frac{2}{3}, \frac{1}{12}]$



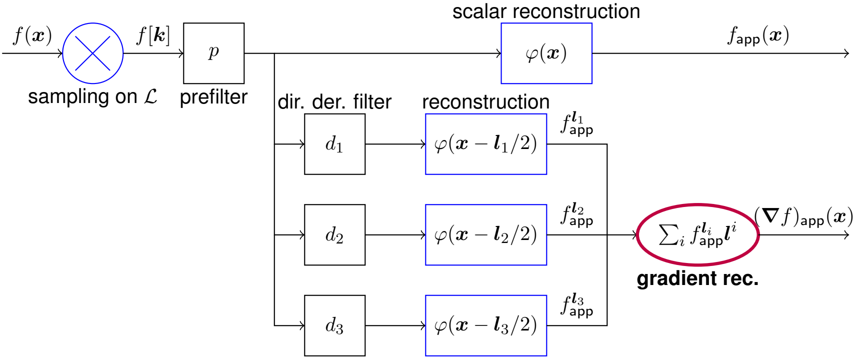
Shifted (*pFIR-s*): $[-\frac{1}{24}, \frac{9}{8}, -\frac{9}{8}, \frac{1}{24}, 0]$

Combined Filter ($p * d$) Comparison

Cubic B-spline



Gradient Reconstruction



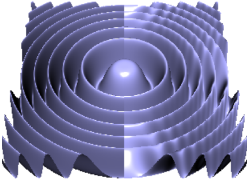
Simple linear transformation

$\boldsymbol{l}^i$: Dual of $\boldsymbol{l}_i$

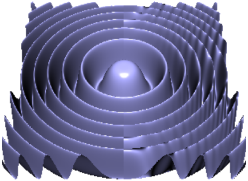
Results

- Tricubic B-splines on the Cartesian Cubic (CC) lattice
- Quintic Box spline on the Body-Centered Cubic (BCC) lattice [Entezari *et al.* ’08]

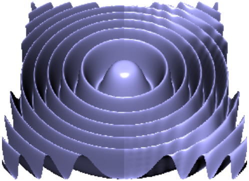
Quantitative Comparison



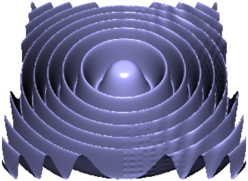
pFIR, $24.14°, 2.11$



pFIR-s, $11.53°, 1.15$



P-OPT26, $18.10°, 1.96$

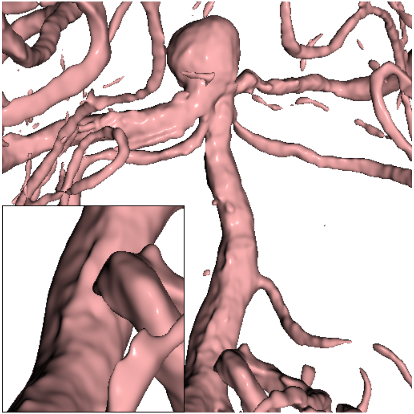


P-FIR-s, $8.7°, 1.03$

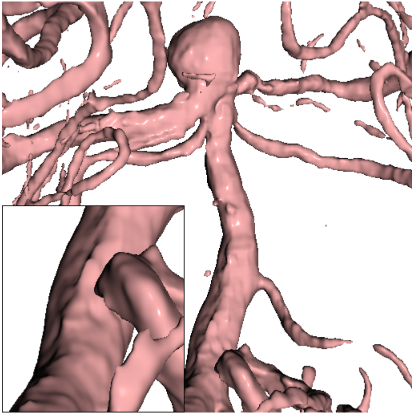
Mean angular and magnitude errors are indicated

Qualitative Comparison

Centered vs. Shifted



pFIR



P-OPT26

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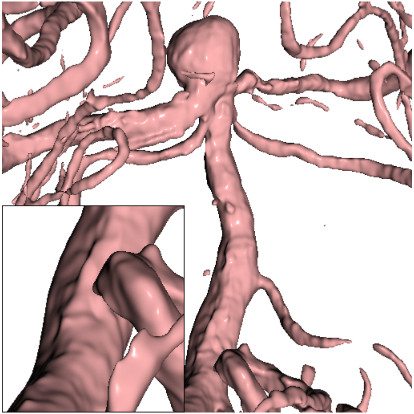

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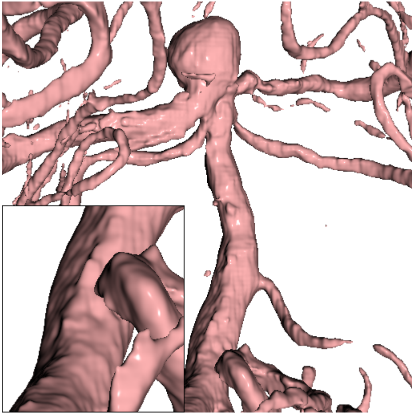
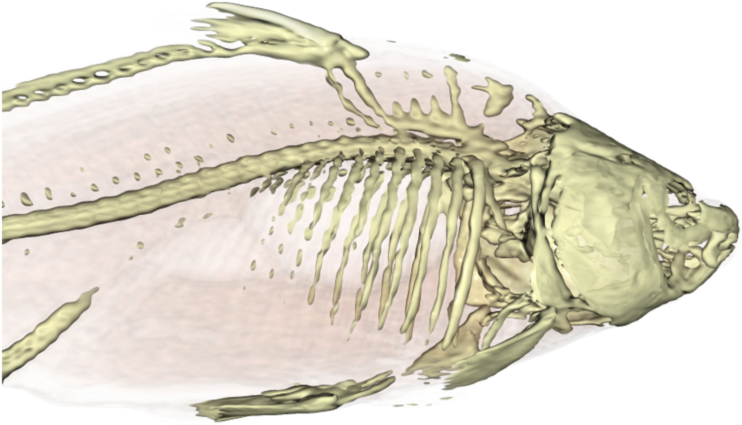


P-FIR-s

Qualitative Comparison

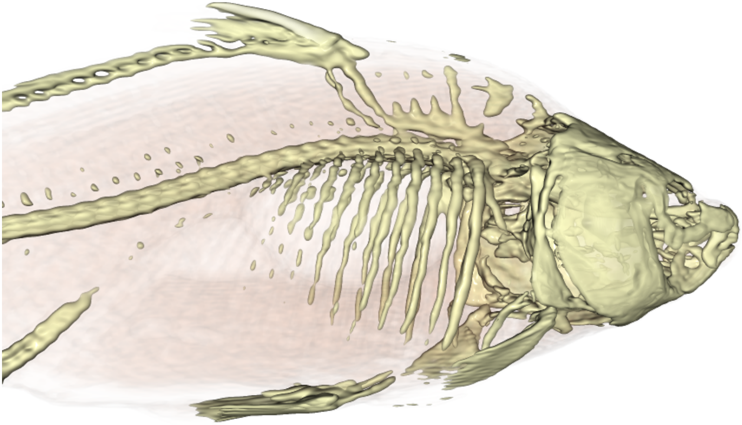
DVR Centered vs. Shifted



pFIR

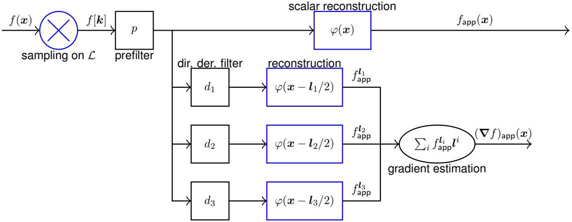
Qualitative Comparison

DVR Centered vs. Shifted



pFIR-s

Conclusion



Contributions

- Error Kernel to quantify accuracy of gradient estimation
- Two frameworks for designing asymptotically optimal derivative filters
- Shifted interpolation function $\rightarrow$ Better quality at no additional cost!

Acknowledgements

NSERC
CRSNG

Thank you for your attention

Contact:

ualim@cs.sfu.ca

Source code is available at:

http://www.cs.sfu.ca/~ualim/personal/research.html

VisIO