VISAGG
Visualization and Graphics Group

UNIVERSITY OF CALGARY

Rendering in Shift-Invariant Spaces

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Department of Computer Science
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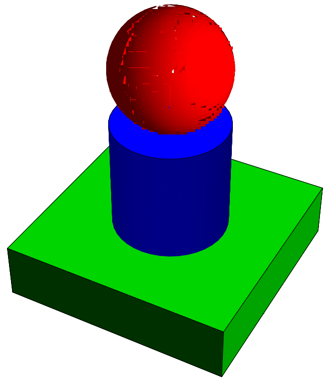
May 31, 2013

Motivation

- Image function describes a continuous 2D image.

$$f : \mathbb{R}^2 \to \mathbb{R}$$



continuous 2D image

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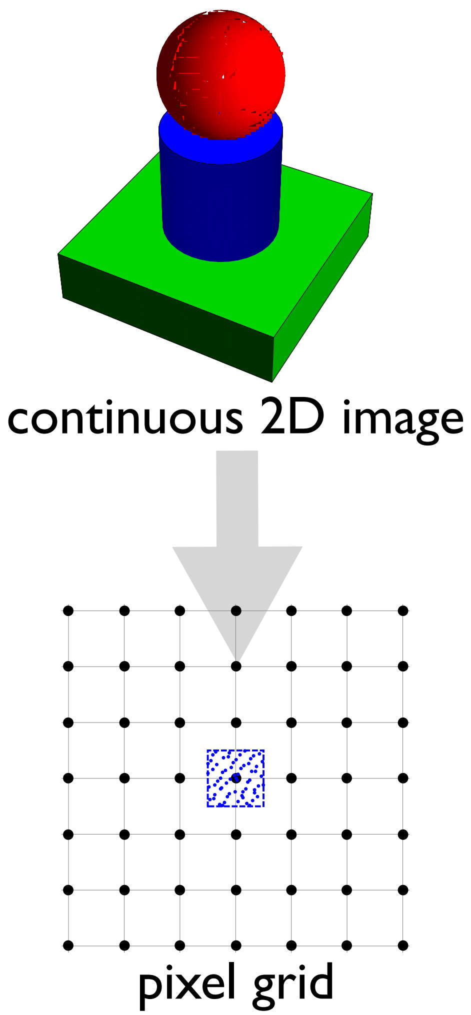


- In rendering, we typically use *area sampling* to discretize the image function f.

Motivation

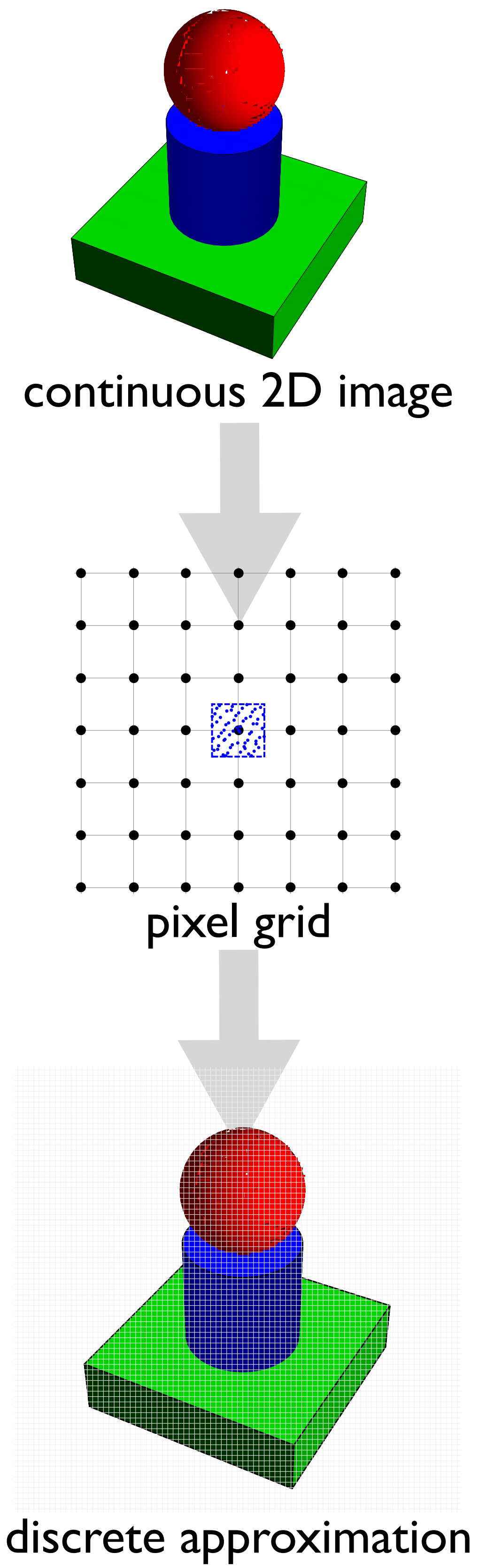
- Image function describes a continuous 2D image.

  $f : \mathbb{R}^2 \to \mathbb{R}$

- In rendering, we typically use *area sampling* to discretize the image function f.

- This yields a discrete approximation of f.

  (can be regarded as continuous in light of Shannon's sampling theory)



Goals

- Use recent developments in signal processing and approximation theory to seek higher-quality grid-based approximations.

- Approximate f in a *shift-invariant space*.

- Seek an approximation f_{app} that minimizes the L_2 error: $\|f - f_{\text{app}}\|_{L_2(\mathbb{R}^2)}$.

- Investigate the quality improvements in the context of ray-tracing.

Outline

- Motivation
- **Shift-Invariant (SI) spaces**
- Rendering in spaces generated by the uniform B-splines
- Results
- Conclusion

Use in Graphics

- More familiar to the image-processing community [Unser 2000]
- Some recent applications in scientific visualization:
  - volume reconstruction [Entezari et al. 2008]
  - gradient estimation [Alim et al. 2010]
- In rendering, antialiasing approaches are largely based on Shannon's sampling theory. SI spaces offer a much more flexible alternative.

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Shift-invariant spaces (1D)

- A vector-space spanned by the (scaled) shifts of a generating function:

$$\mathbb{V}(\varphi) := \left\{ g(x) = \sum_{n \in \mathbb{Z}} c[n]\varphi(x-n) : c \in l_2(\mathbb{Z}) \right\}$$

Inner product:
$\langle f, g \rangle := \int_{\mathbb{R}} f(x) g(x) dx$

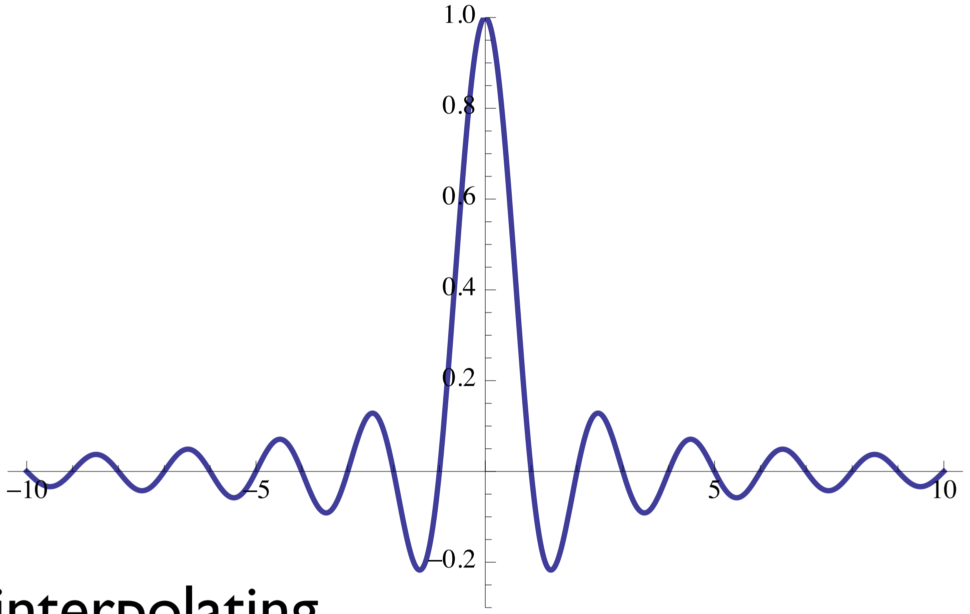
- The sequence $\{\varphi(x-n)\}_{n\in\mathbb{Z}}$ forms a basis for $\mathbb{V}(\varphi)$.
  (The basis may not be orthogonal)

- There exists a *dual basis* $\{\mathring{\varphi}(x-n)\}_{n\in\mathbb{Z}}$ that also spans $\mathbb{V}(\varphi)$. Primal and dual generators are bi-orthonormal:
  $\langle \varphi, \mathring{\varphi}(\cdot - n) \rangle = \delta[n]$

Familiar Examples

- Shannon's sampling theory is a special case.

$$\varphi(x) = \mathrm{sinc}(x)$$



- interpolating
- infinite support (not practical)
- self-dual
- can represent all bandlimited functions

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Familiar Examples

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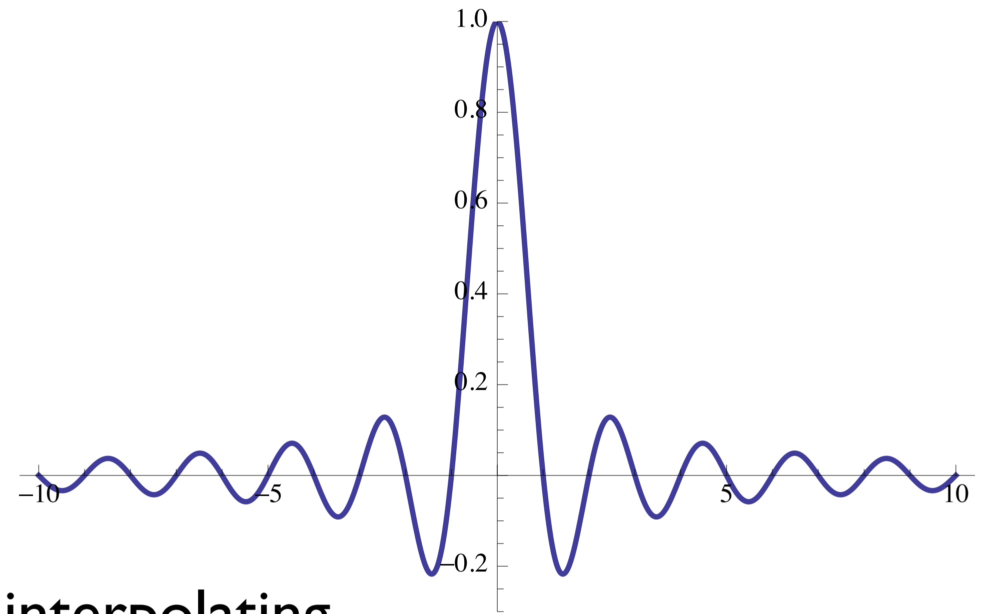

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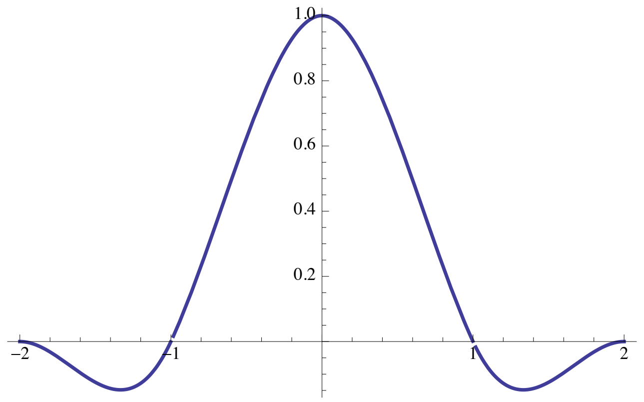
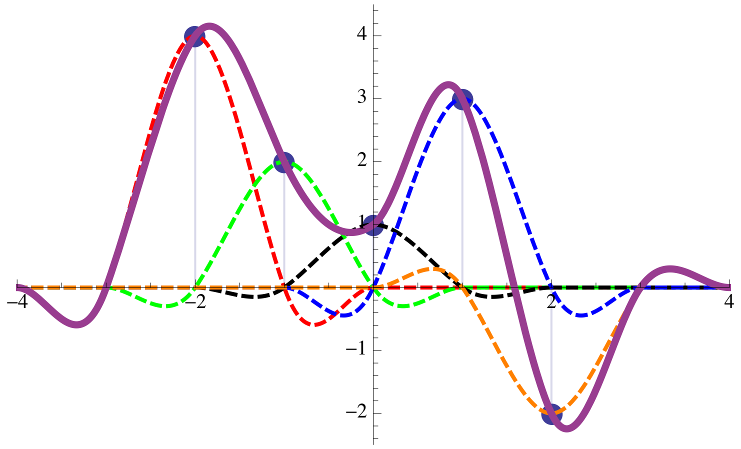
- Cubic interpolation [Mitchell Netravali 88]

$$\varphi(x)$$





- interpolating
- compact support (practical)
- primal and dual are piecewise cubic
- can represent all cubic polynomials

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Minimum-error Approximation

- Minimum-error approximation is obtained by orthogonally projecting f to the desired space $\mathbb{V}(\varphi)$.

- Image approximation:

$$f_{\mathsf{app}}(x) = \sum_n c[n]\varphi(x-n).$$

- Measure the coefficients according to:

$$\boxed{c[n] = \langle f, \mathring{\varphi}(\cdot - n)\rangle}$$

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(the dual $\mathring{\varphi}$ is usually not compactly supported)

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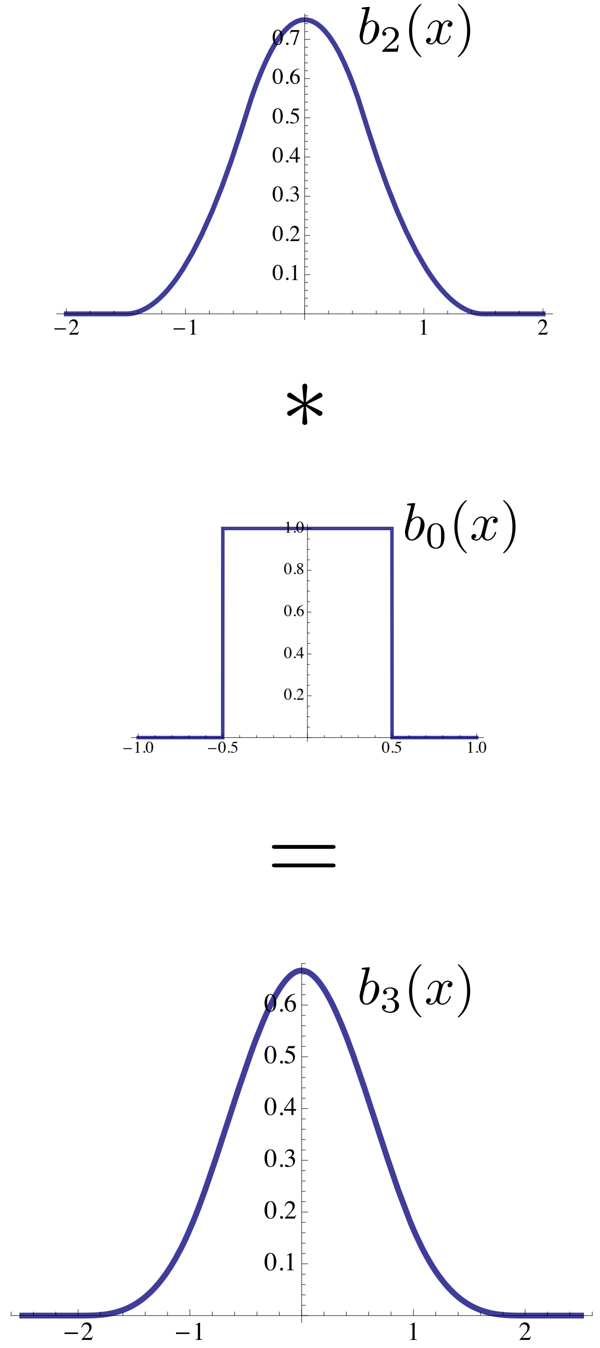
Uniform Centered B-splines

- The uniform centered B-spline $b_d(x)$ (d-degree) is obtained via d convolutions of the box function.

$$
\begin{aligned}
b_d(x) :&= \underbrace{(b * b * \ldots * b)}_{d+1 \text{ repetitions of } b}(x) \\
&= (b * b_{d-1})(x)
\end{aligned}
$$

- For example, the cubic B-spline can be obtained in two different ways:



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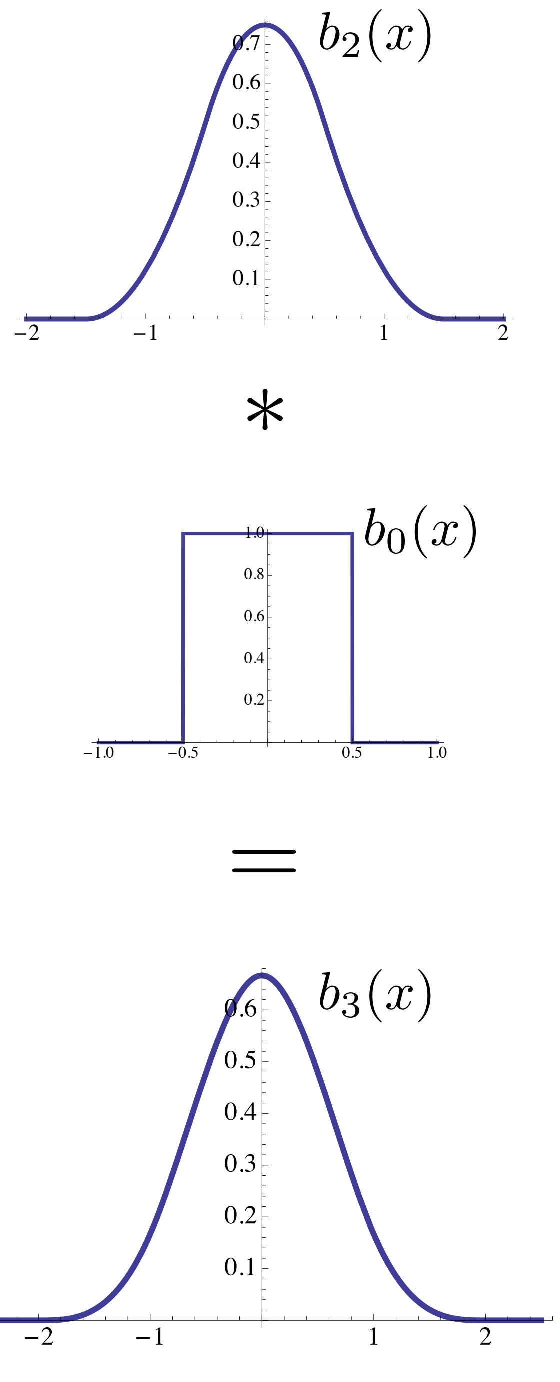

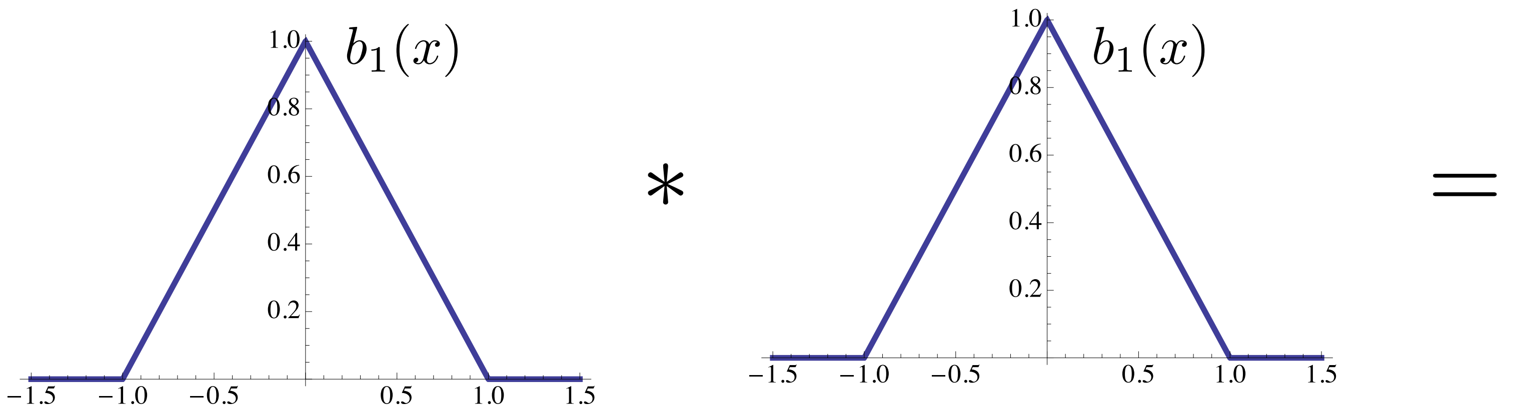

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Dual Generators

- The box function is self-dual. Higher degree B-splines are not.

- Duals have the following Fourier-domain expression:

$$\hat{\mathring{\varphi}}(\omega) = \frac{\hat{\varphi}(\omega)}{\hat{a}_{\varphi}(\omega)}$$

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DTFT of the autocorrelation sequence.
Can be computed analytically for the B-splines.

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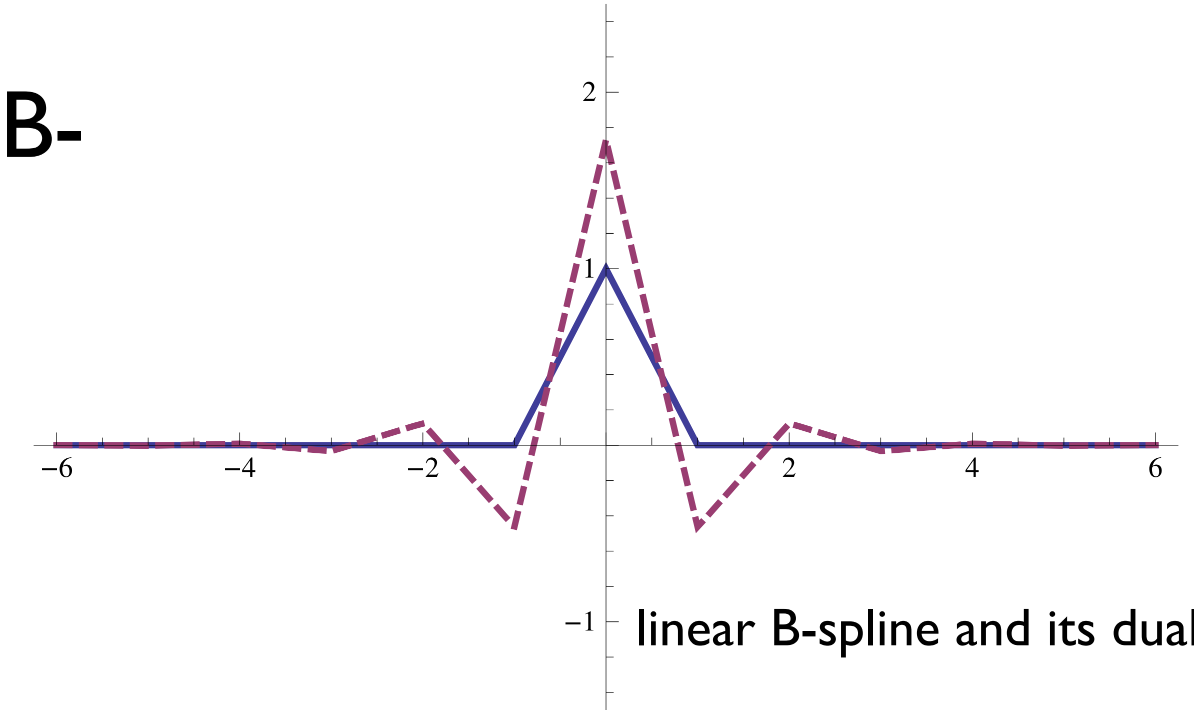

linear B-spline and its dual

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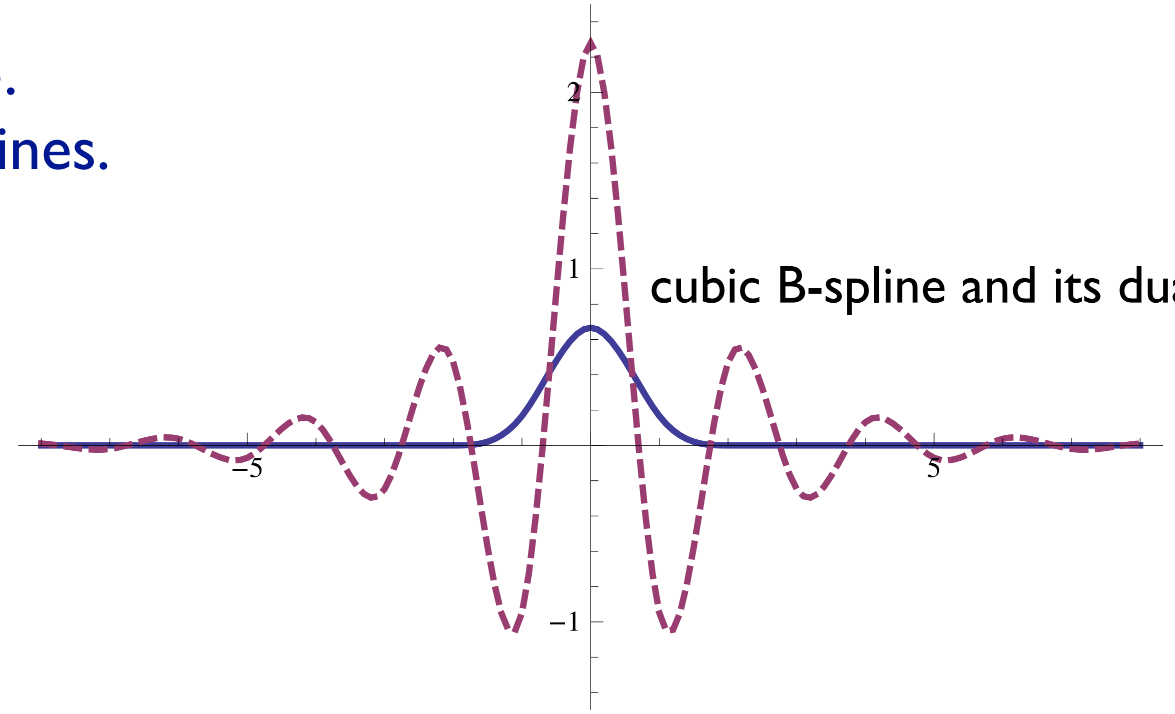

cubic B-spline and its dual

Rendering

- Recall that we wish to compute:

$$\boxed{c[n] = \langle f, \mathring{\varphi}(\cdot - n)\rangle}$$

- Replacing the dual with its primal representation, we get:

$$\begin{aligned} c[n] &= \sum_m a_\varphi^{-1}[m] \left\langle f, \varphi\big(\cdot - (n-m)\big)\right\rangle \\ &= (r * a_\varphi^{-1})[n] \end{aligned}$$

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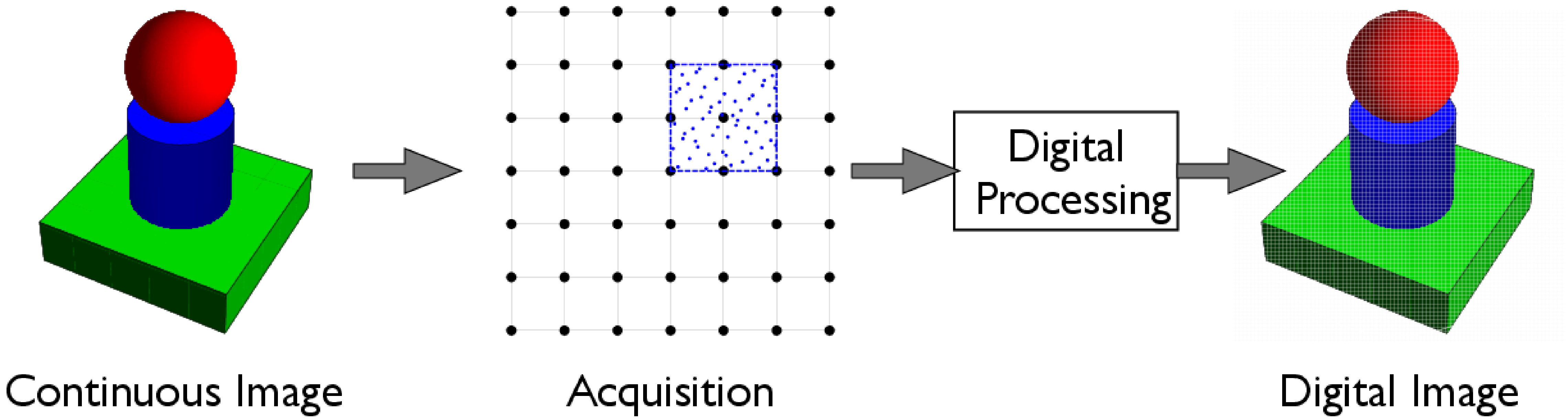
analog acquisition digital processing

Rendering

$$f_{\mathsf{app}}(x) = \sum_n (r * a_\varphi^{-1})[n]\varphi(x-n)$$



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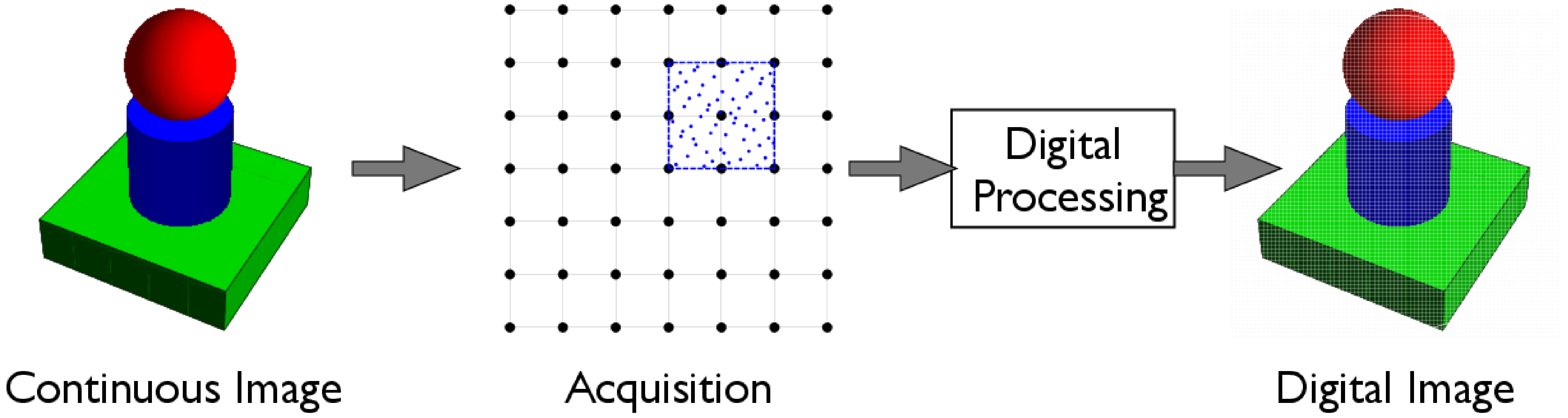
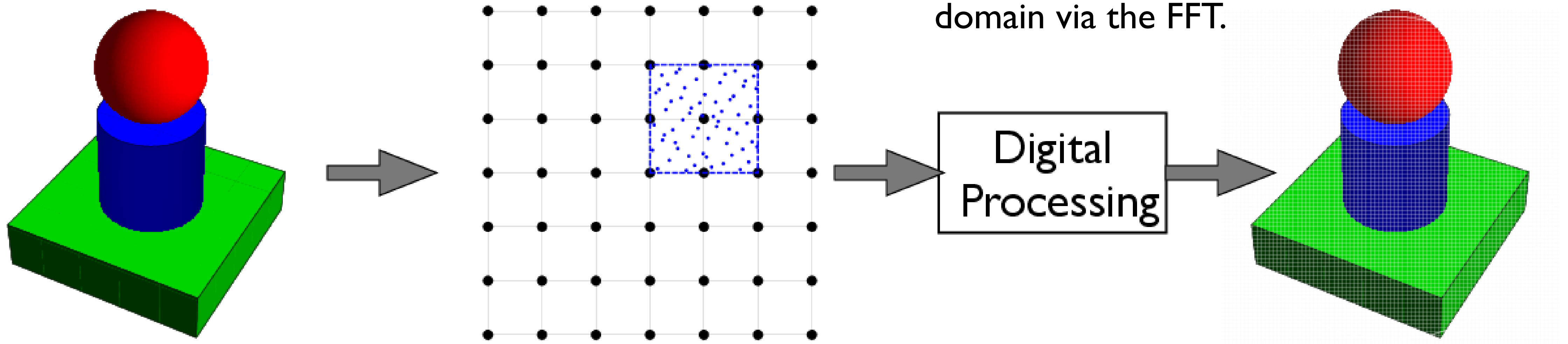


- Traces several rays through the support of the shifted generator.
- AKA antialiasing, smooths out high frequencies.

Rendering

$$f_{\text{app}}(x) = \sum_n (r * a_\varphi^{-1})[n]\varphi(x-n)$$

- Convolve with the inverse autocorrelation sequence.
- Restores some high frequencies.
- Efficiently implemented in the Fourier domain via the FFT.



Continuous Image

Acquisition

Digital Image

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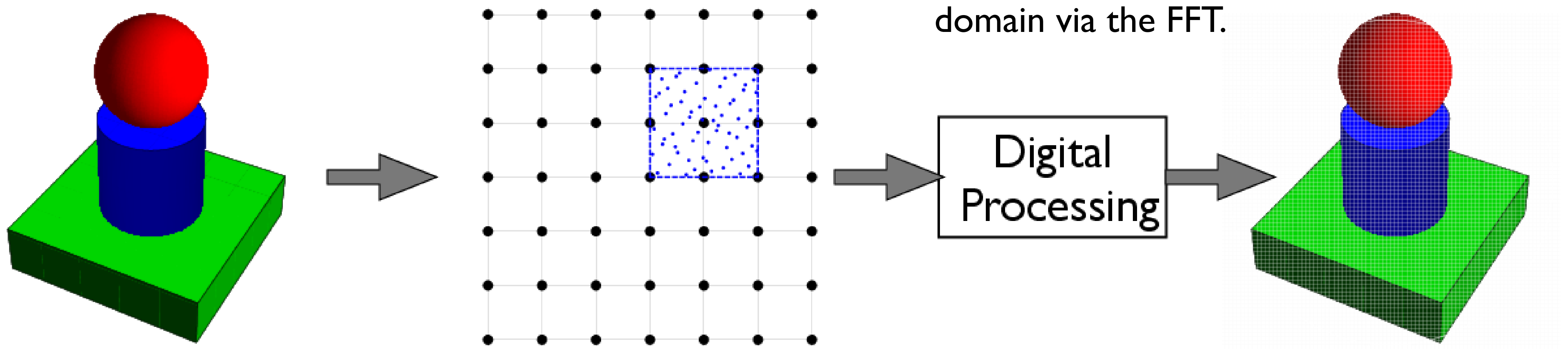

Continuous Image

Acquisition

- Traces several rays through the support of the shifted generator.
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Digital Image

- In most cases, simply sample f_{app} at the integers to obtain the final image.

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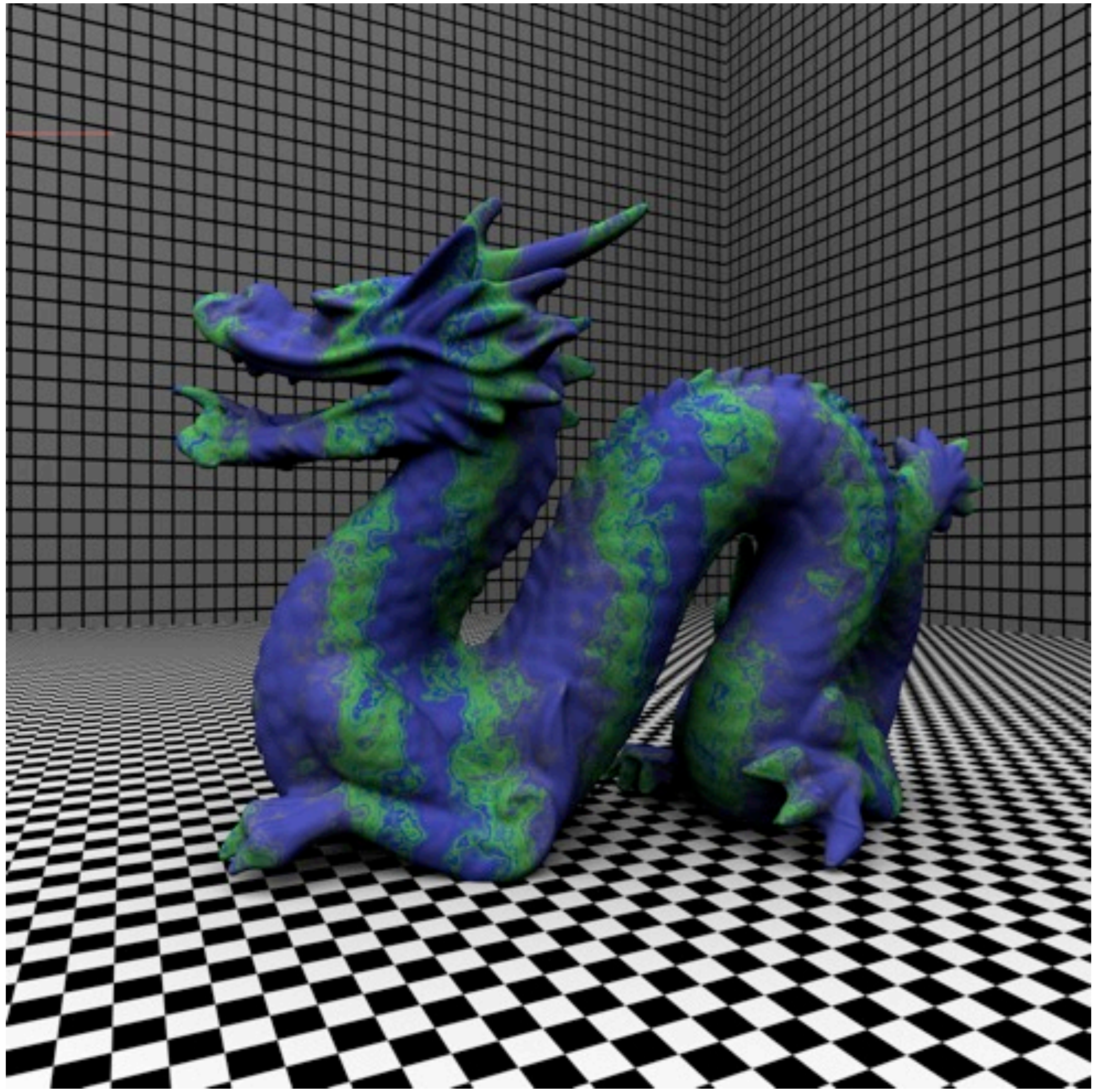
Test Scene



Existing	OP
conv. tent	min. error tent
MN	min. error cubic

Experimental setup:

- Low discrepancy sampler (256 rays per coefficient)
- Rendered at a resolution of 1000 x 1000
- Low dynamic range

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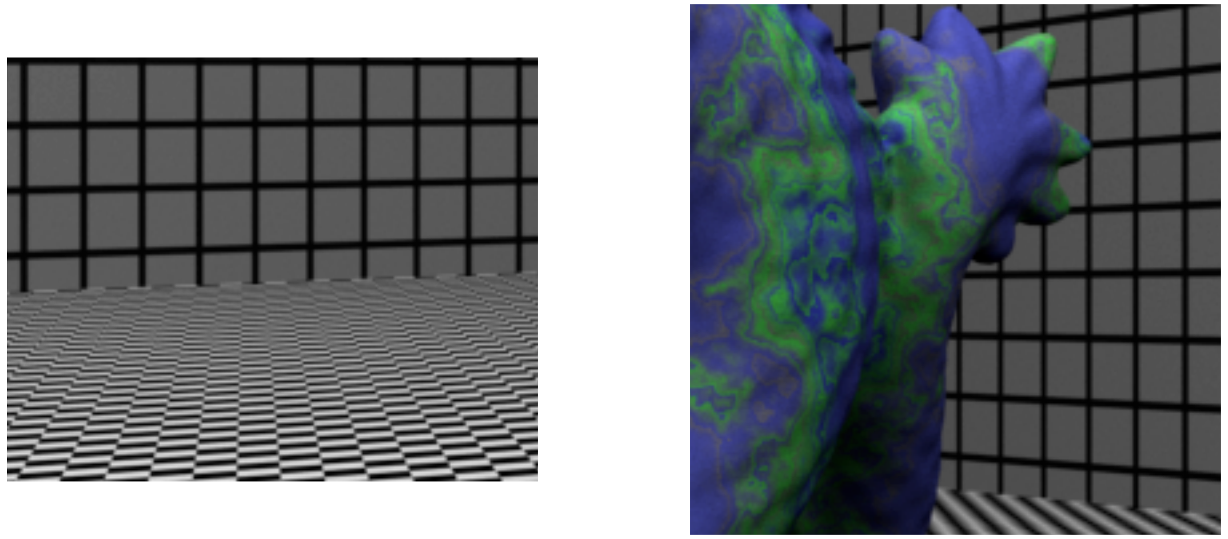
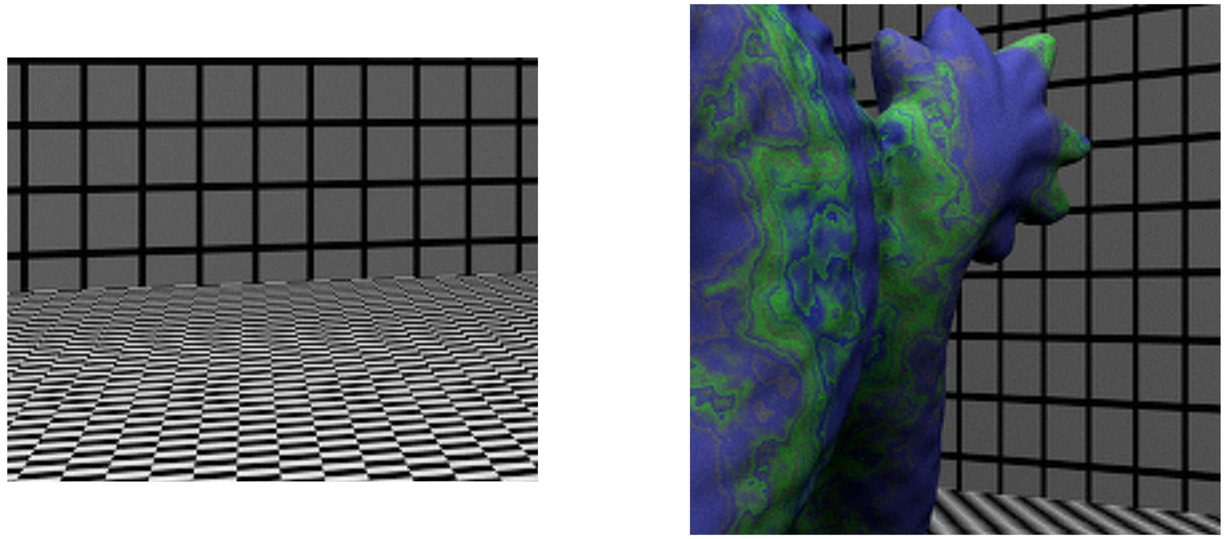
Linear Generators

conv. tent



min. error tent



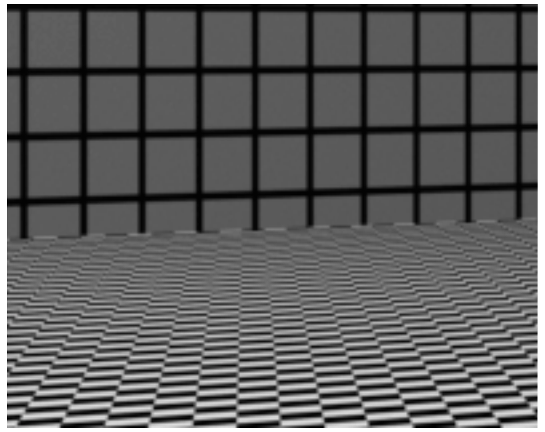
- Effective anti-aliasing
- Smooths out detail in other areas

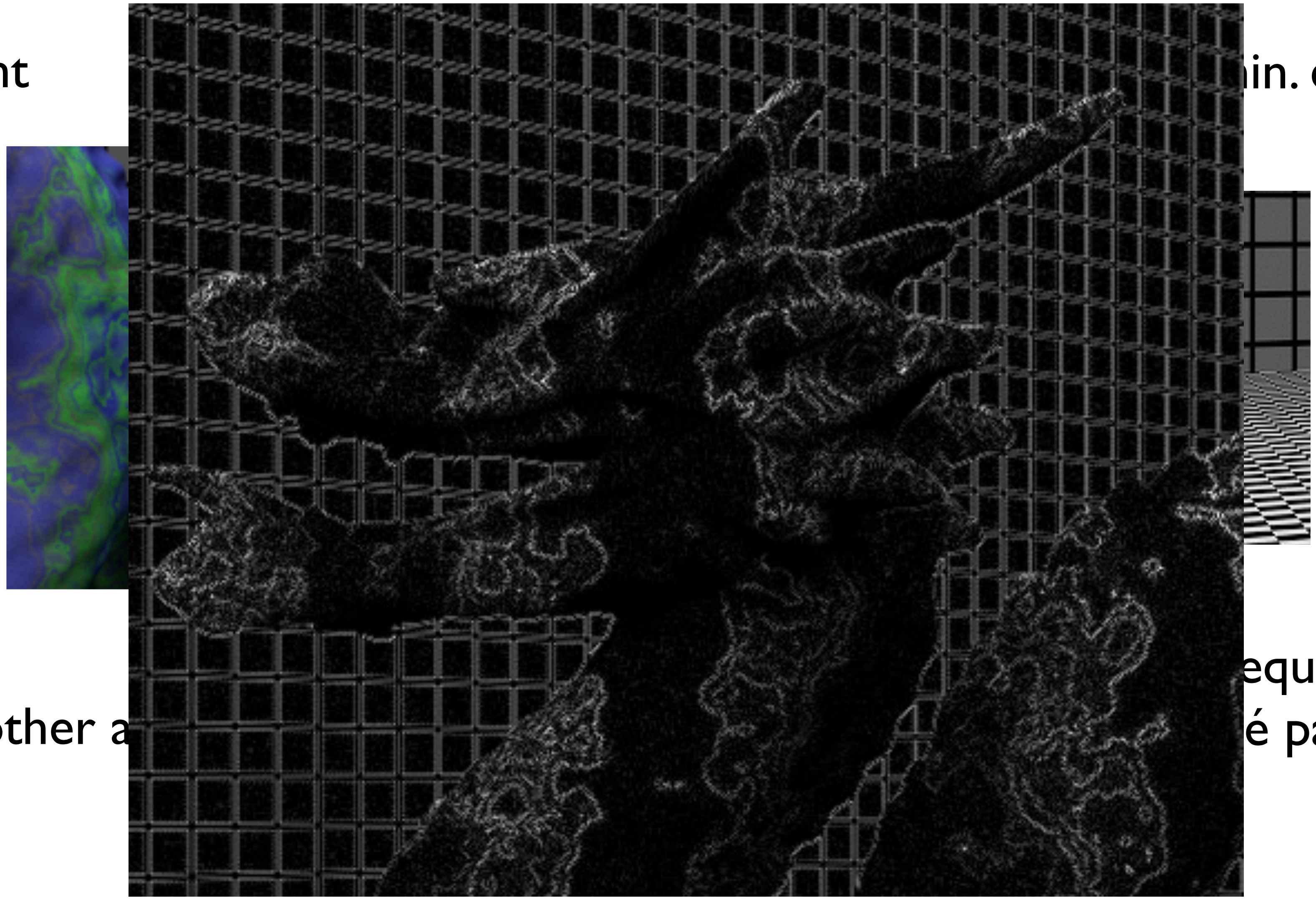
- Restores high frequency detail
- Introduces Moiré patterns

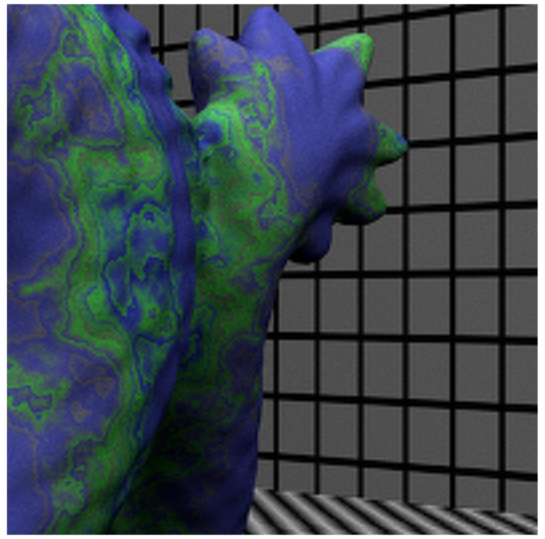
Linear Generators

conv. tent

in. error tent







- Effective anti-aliasing
- Smooths out detail in other a

equency detail

é patterns

difference image

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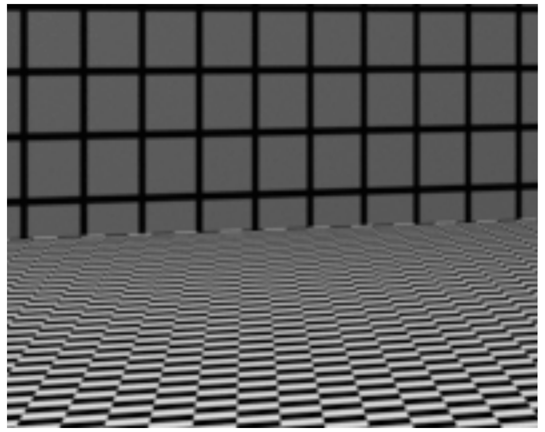
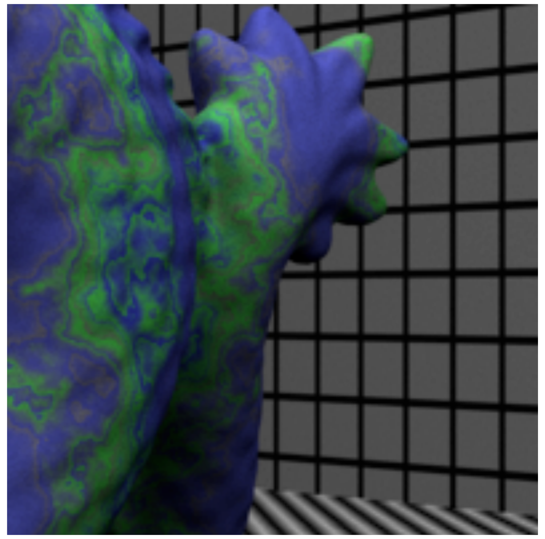
Cubic Generators

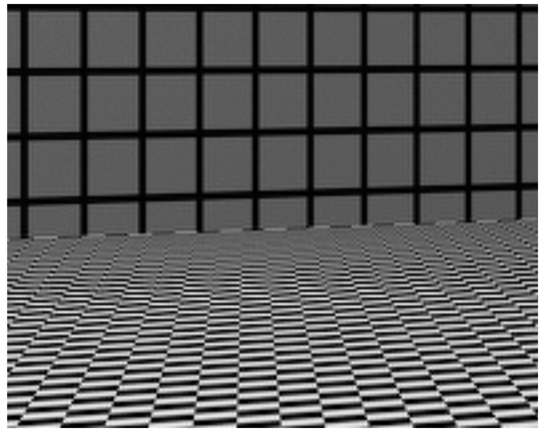
MN

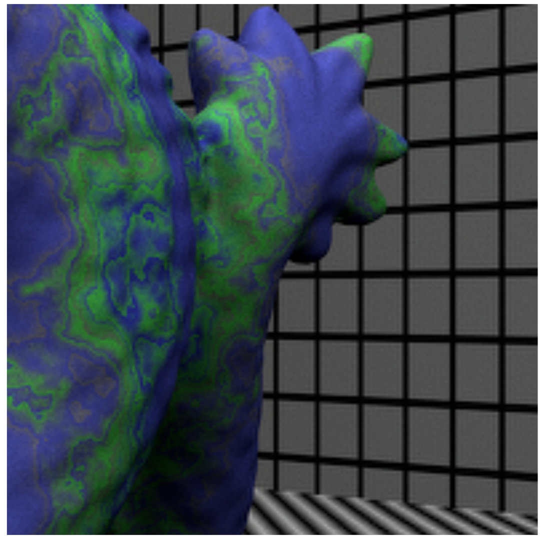




- Effective anti-aliasing
- Slightly sharper than conv. tent

min. error cubic



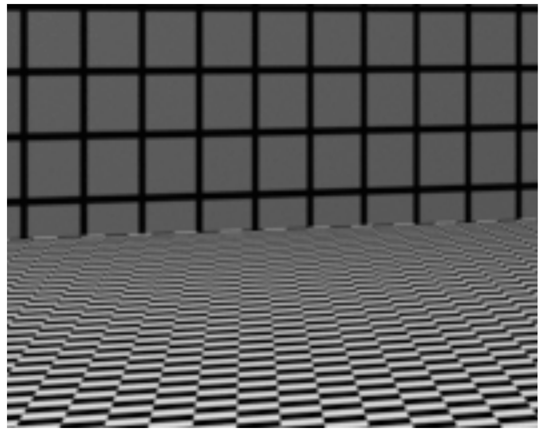


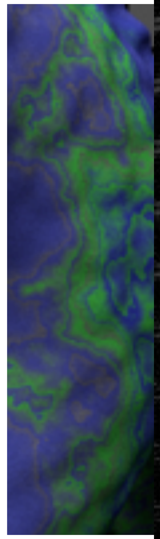
- Reduces Moiré patterns
- Slightly blurrier than min. error tent

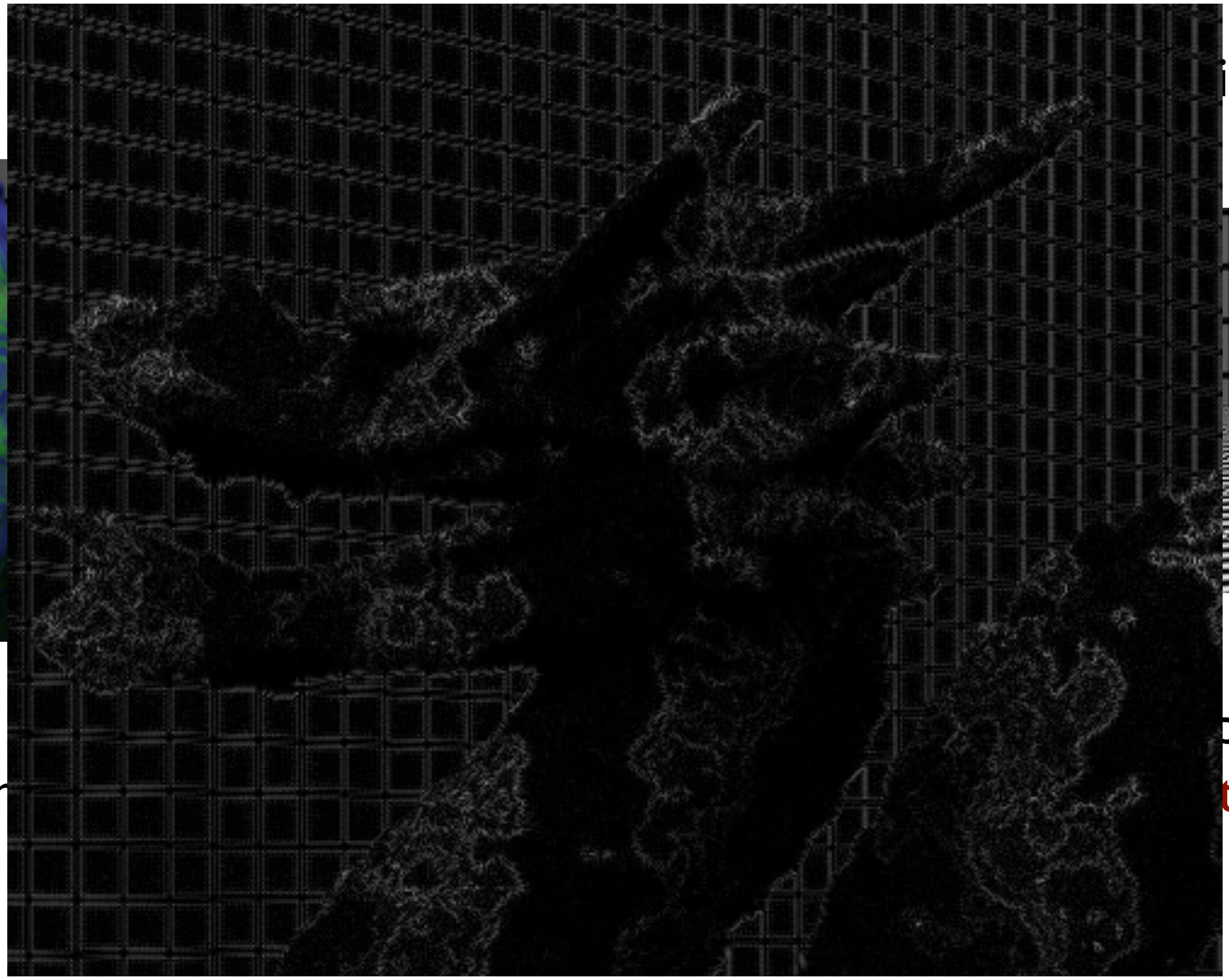
Cubic Generators

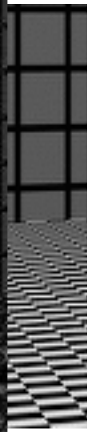
MN

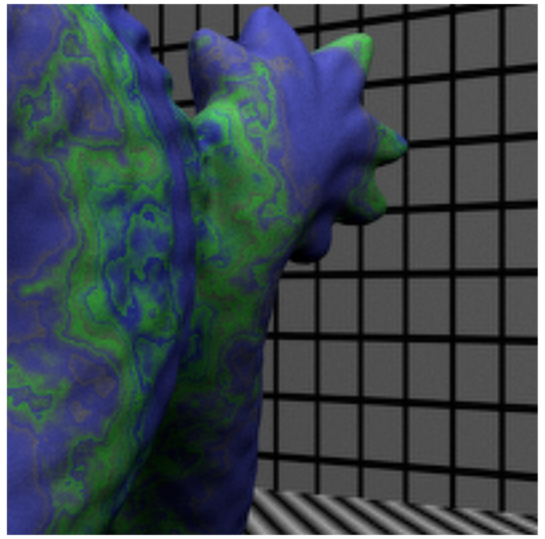
in. error cubic











- Effective anti-aliasing
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patterns

than min. error tent

difference image

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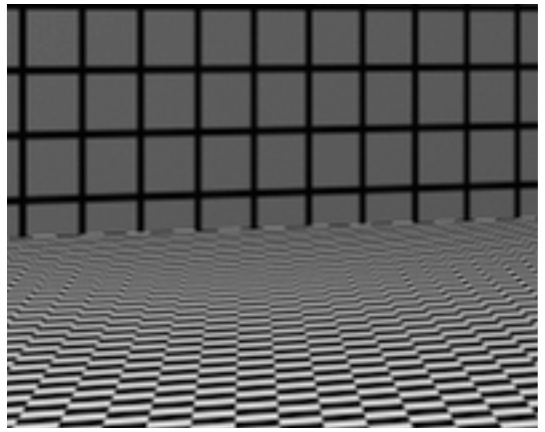
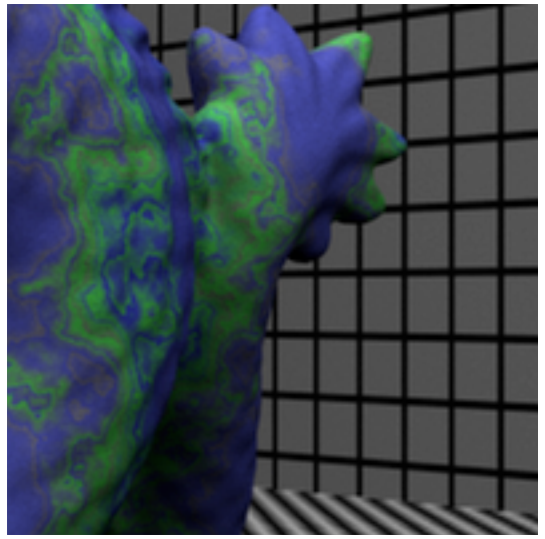
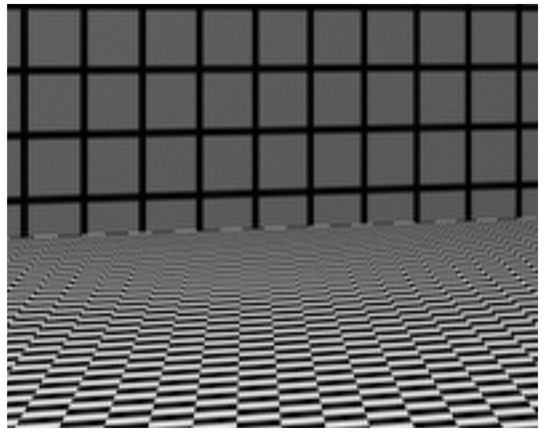
Removing Moiré Patterns

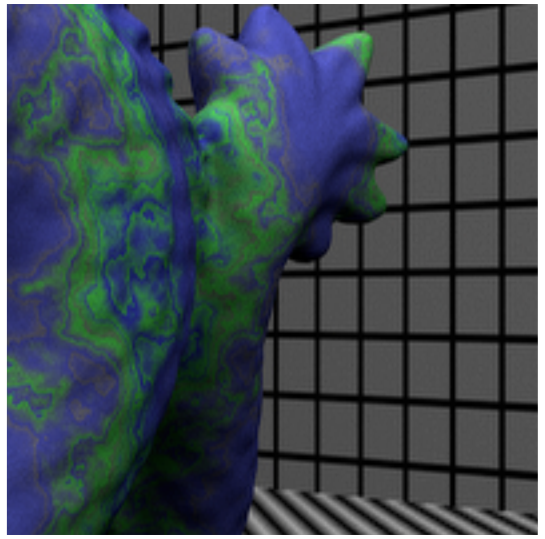
min. error tent





min. error cubic





- Since $f_{\text{app}}(x)$ is continuous, it can be sampled at off-center points to diminish Moiré patterns.
- Effective antialiasing without unduly sacrificing sharpness.

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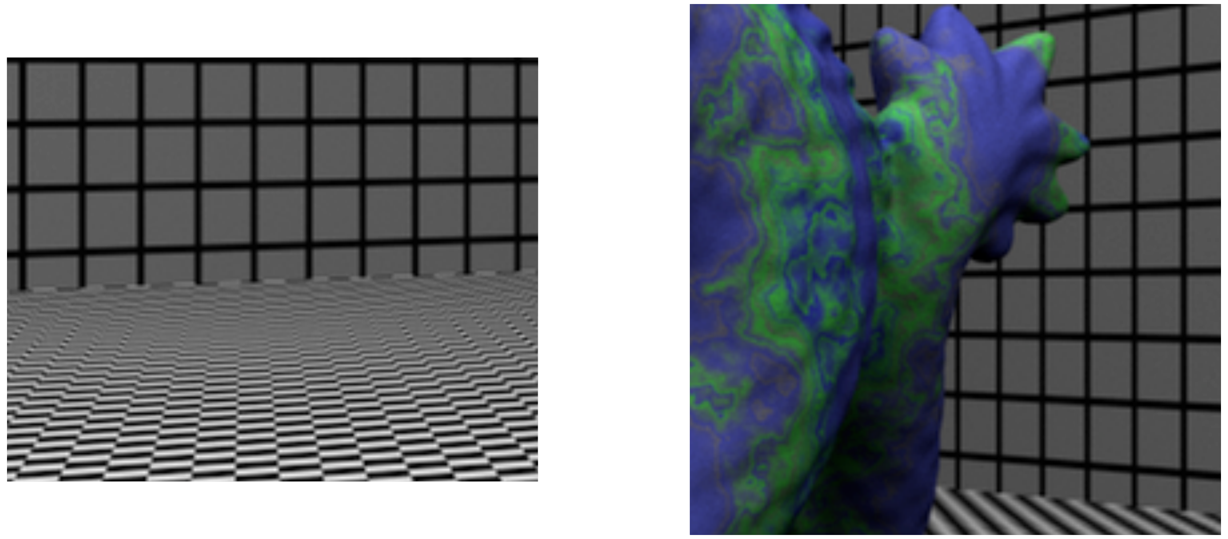
Removing Moiré Patterns

min. error tent



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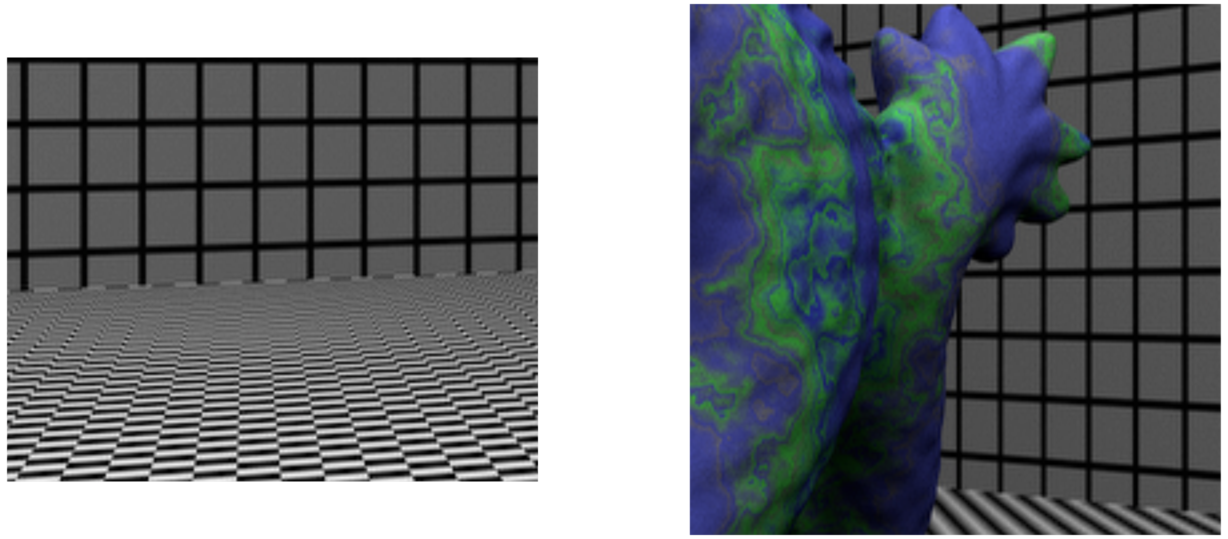

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Filter	Percentage
Box	16.56
Conv. tent	15.19
Min-error tent	**19.18**
Mitchell-Netravali	15.50
Min-error cubic	17.41
Min-error tent (AA)	15.97
Min-error cubic (AA)	**16.34**

Energy percentage (measured via the FFT) that lies in the high-pass regime

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Not-so-synthetic Scenes

- Scenes that do not contain pathologically high frequencies have a clear advantage.

conv. tent

min. error tent

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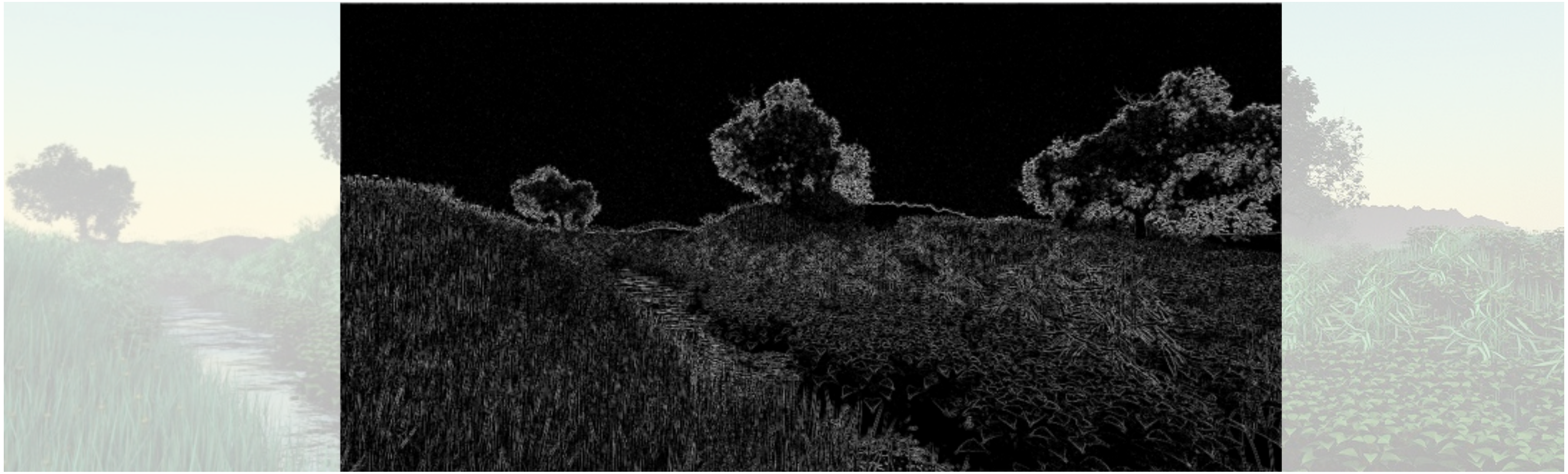

conv. tent | difference image | min. error tent

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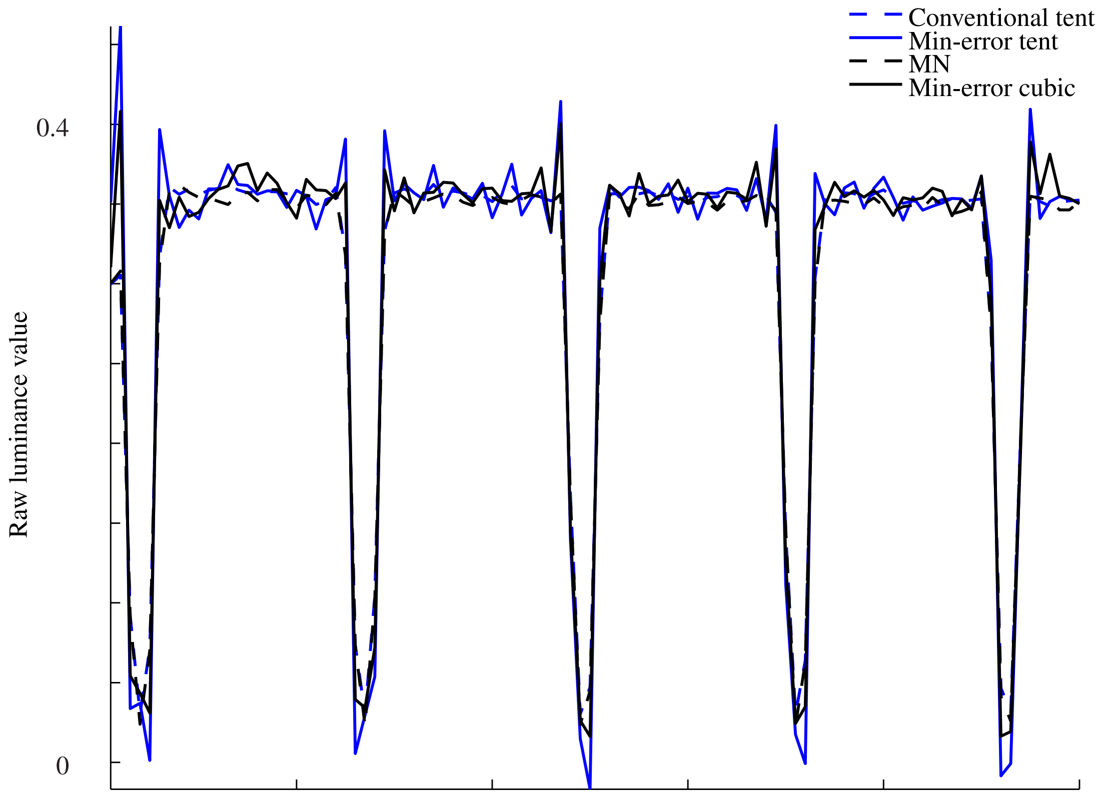
Limitations

- Currently limited to LDR images. Further analysis is needed to extend to HDR images.

- Min. error approximation recovers high frequencies. Therefore, a relatively noise-free acquisition step is necessary.



Min. error approximation leads to overshoots and undershoots across discontinuities.

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Future Work

- Extend to hexagonal grids, dynamic scenes
- Multiresolution
- Exploit sparsity, compressive sensing

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Future Work

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- Multiresolution
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Thank you for your attention

ualim@ucalgary.ca
http://visagg.cpsc.ucalgary.ca