# Operator Discretization in Shift-Invariant Spaces

## ICOSAHOM 2014

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# Outline

# Sampling Lattices

Generated by taking integer combinations of columns of $\mathbf{L}$, i.e. $\mathcal{L} = \mathbf{L}\boldsymbol{k}$ where $\boldsymbol{k} \in \mathbb{Z}^s$.

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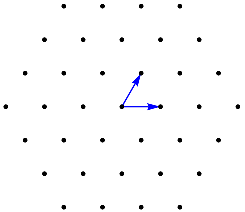

2D hexagonal lattice

$$\mathbf{L} = \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} \end{bmatrix}$$

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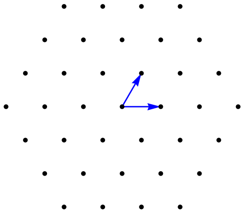

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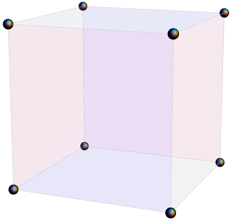
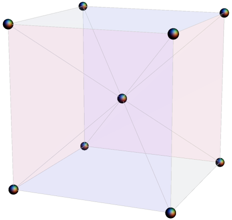


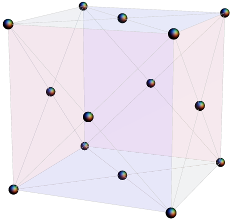
CC lattice

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



BCC lattice

$$\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$



FCC lattice

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

# Function Approximation

## Shift-Invariant Spaces

$$\mathbb{V}(\mathcal{L}_h, \varphi) := \left\{ g(\boldsymbol{x}) = \sum_{\boldsymbol{n} \in \mathbb{Z}^s} c[\boldsymbol{n}] \varphi_{h,\boldsymbol{n}}(\boldsymbol{x}) : c \in l_2(\mathbb{Z}^s) \right\}, \quad \varphi_{h,\boldsymbol{n}}(\boldsymbol{x}) := \varphi(\frac{\boldsymbol{x}}{h} - \mathbf{L}\boldsymbol{n}).$$

- Find $f_{\text{app}} \in \mathbb{V}(\mathcal{L}_h, \varphi)$ that *attempts* to minimize the $L_2$-error $\|f - f_{\text{app}}\|$.
- $\varphi$ can be $\mathrm{sinc}$-*like* (infinite support) or *spline-like* (compact support).

# Outline

# Lattice Isotropy (2D)

For isotropically *bandlimited* functions, the optimal sampling lattice $\mathcal{L}$ is the one whose dual $\mathcal{L}^\circ$ is the optimal sphere-packing lattice [Petersen and Middleton, 1962, Lu et al., 2009].

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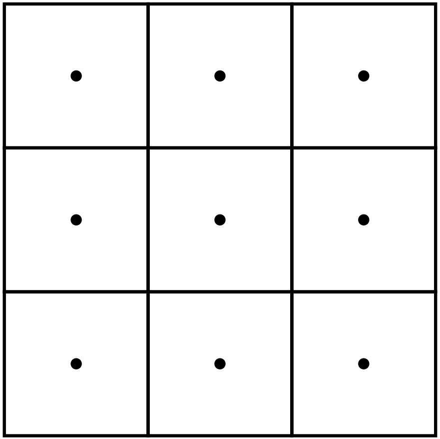

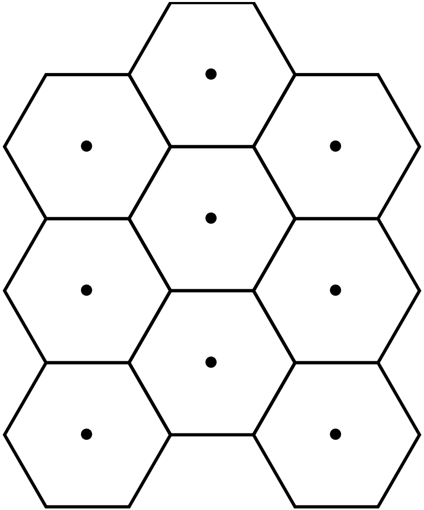

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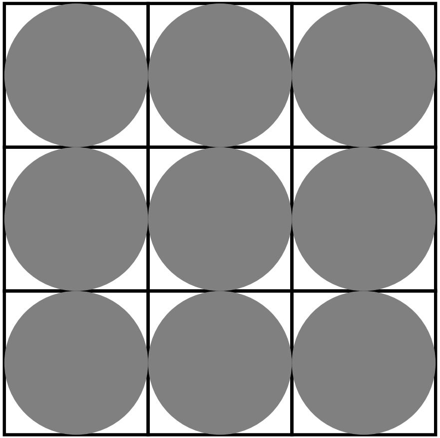
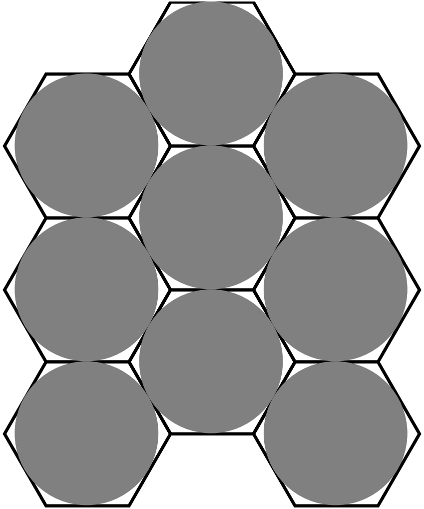


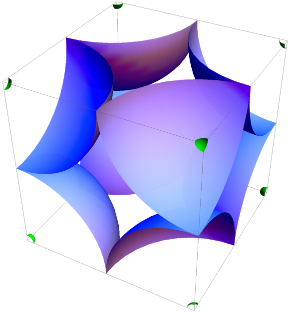
Cartesian ($78.5\%$ efficient)



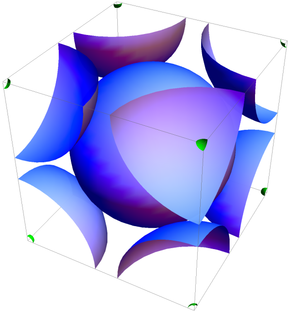
hexagonal ($90.6\%$ efficient)

# Lattice Isotropy (3D Cubic Lattices)

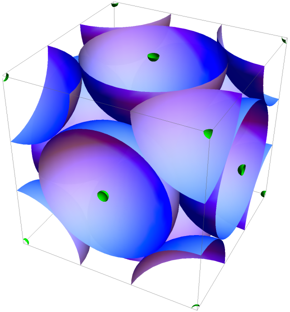
- The FCC lattice is the optimal sphere-packing lattice [Conway and Sloane, 1999].
- The BCC lattice is the optimal sampling lattice.



CC (52%)



BCC (68%)



FCC (**74%**)

# Error Kernel

- **Measurement Model:**
  Discrete measurements made according to $c[\boldsymbol{n}] = \langle f, \tilde{\varphi}_{h,\boldsymbol{n}} \rangle$.
  ($\tilde{\varphi}$ is an analysis function)

**Fourier Error Kernel (extension of [de Boor et al., 1994, Blu and Unser, 1999])**

$$\|f - f_{\text{app}}\|^2 = \int_{\mathbb{R}^s} \left|\hat{f}(\boldsymbol{\omega})\right|^2 E(h\boldsymbol{\omega}) d\omega,$$

$$\text{where} \quad E(\boldsymbol{\omega}) := \underbrace{1 - \frac{\left|\hat{\varphi}(\boldsymbol{\omega})\right|^2}{\hat{A}_\varphi(\boldsymbol{\omega})}}_{E_{\text{min}}(\boldsymbol{\omega})} + \underbrace{\hat{A}_\varphi(\boldsymbol{\omega})\left|\hat{\tilde{\varphi}}(\boldsymbol{\omega}) - \hat{\mathring{\varphi}}(\boldsymbol{\omega})\right|^2}_{E_{\text{res}}(\boldsymbol{\omega})}.$$

- $a_\varphi[\boldsymbol{n}] \leftrightarrow \hat{A}_\varphi(\boldsymbol{\omega})$ is the autocorrelation sequence of $\varphi$.
- $\mathring{\varphi}$ is the biorthogonal dual of $\varphi$.

# Error Kernel

## Fourier Error Kernel

$$E(\boldsymbol{\omega}) := \underbrace{1 - \frac{\left|\hat{\varphi}(\boldsymbol{\omega})\right|^2}{\hat{A}_\varphi(\boldsymbol{\omega})}}_{E_{\text{min}}(\boldsymbol{\omega})} + \underbrace{\hat{A}_\varphi(\boldsymbol{\omega})\left|\hat{\tilde{\varphi}}(\boldsymbol{\omega}) - \hat{\mathring{\varphi}}(\boldsymbol{\omega})\right|^2}_{E_{\text{res}}(\boldsymbol{\omega})}.$$

- For an orthogonal projection, $\tilde{\varphi} = \mathring{\varphi}$ and $E_{\text{res}}(\boldsymbol{\omega}) = 0$.
- $E_{\text{min}}(\boldsymbol{\omega}) = O(\|\boldsymbol{\omega}\|^{2k})$ where $k$ is the approximation order provided by $\varphi$, i.e. $\|f - f_{\text{app}}\| = O(h^k)$.
- For suboptimal approximations (e.g. when $f$ is point-sampled), the goal is to design $\tilde{\varphi}$ so that $E_{\text{res}} = O(\|\boldsymbol{\omega}\|^{2k})$ (e.g. interpolation, quasi-interpolation).

# Outline

# Minimum Error Comparison: 3D Cubic Lattices

- Approximating a $jinc$ function. Nearest neighbour on CC, BCC, and FCC. Trilinear and tricubic B-spline on CC vs linear and quintic box spline on BCC.

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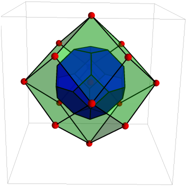
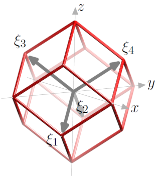


Voronoi cells of BCC and FCC lattices




support of linear box spline on BCC (image courtesy of [Entezari et al., 2008])

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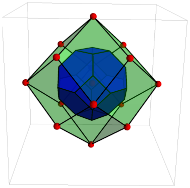

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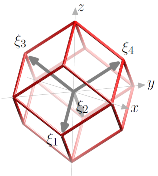


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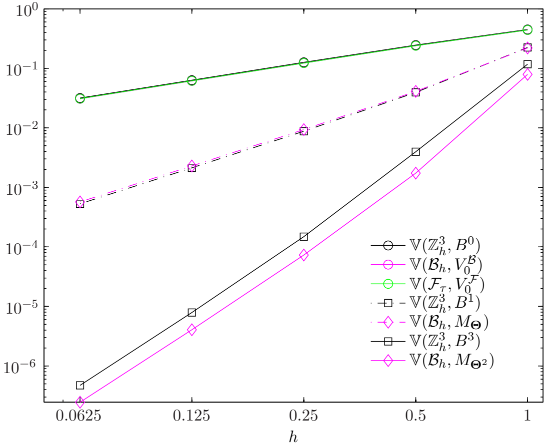


(a) Oversampled

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# Motivation

- Shading in volume visualization, poor gradients lead to poor visuals.

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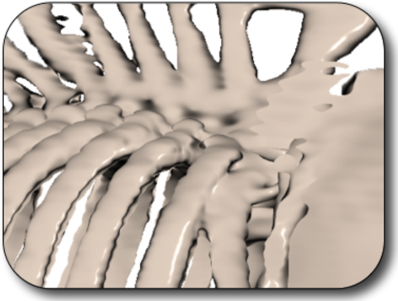

central differencing

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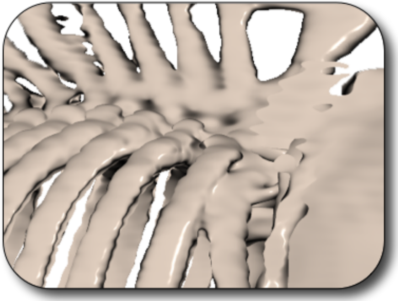
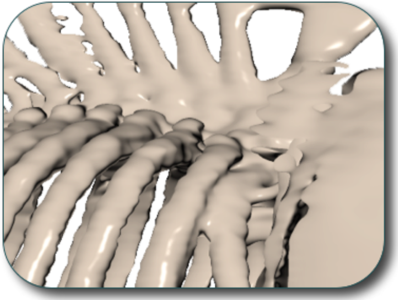


central differencing



orthogonal projection

# Overview

## Problem

Given samples $f[\boldsymbol{n}] = f(h\mathbf{L}\boldsymbol{n})$ of $f$, estimate the gradient $\boldsymbol{\nabla} f$.

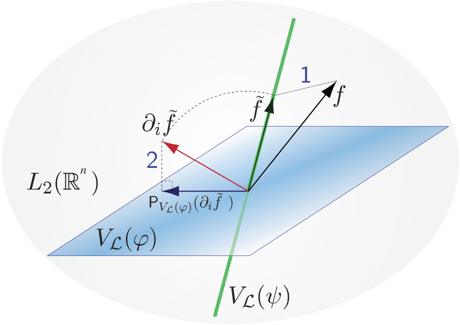
## Summary of Approach

- Seek independent approximations $f^i_{\text{app}} \approx \partial_i f$ such that $f^i_{\text{app}} \in \mathbb{V}(\mathcal{L}_h, \varphi)$.
- Discrete filtering:

$$f^i_{\text{app}}(\boldsymbol{x}) = \frac{1}{h} \sum_{\boldsymbol{n}} (f * q_i)[\boldsymbol{n}] \varphi_{h,\boldsymbol{n}}(\boldsymbol{x}).$$

- First approximate $f$ in an auxiliary space $\mathbb{V}(\mathcal{L}_h, \psi)$, then project the derivative of the auxiliary approximation to the target space $\mathbb{V}(\mathcal{L}_h, \varphi)$.

# Two-stage Approximation Model




$$q_i[\boldsymbol{n}] = (\underbrace{p_1}_{\text{Interp.}} * \underbrace{\mathring{d}_i}_{\text{Proj.}})[\boldsymbol{n}]$$

1. $p_1[\cdot] \leftrightarrow \hat{P}_1(\boldsymbol{\omega}) = \left(\sum_{\boldsymbol{k}} \psi(\mathbf{L}\boldsymbol{n}) \exp(-2\pi\imath\boldsymbol{\omega}^{\mathsf{T}}\boldsymbol{k})\right)^{-1}$.
2. $\mathring{d}_i[\boldsymbol{n}] := \left\langle \partial_i \psi, \mathring{\varphi}_{1,\boldsymbol{n}} \right\rangle$.

Filters are expensive!

# Results: Trilinear Interpolation (CC)

(a) *2-cd* (b) *ll* (c) *ql*

# Outline

# A More Flexible Approach

## Problem

Given samples $f[\boldsymbol{n}] = f(h\mathbf{L}\boldsymbol{n})$ of $f$, estimate the gradient $\boldsymbol{\nabla} f$.

## Summary of Approach

- Seek independent approximations $f_{\text{app}}^{\boldsymbol{l}_i} \approx (\boldsymbol{\nabla} f) \cdot \boldsymbol{l}_i$ such that $f_{\text{app}}^{\boldsymbol{l}_i} \in \mathbb{V}(\mathcal{L}_h, \varphi_i)$, where $\varphi_i(\boldsymbol{x}) := \varphi(\boldsymbol{x} - \frac{\boldsymbol{l}_i}{2})$ and $\boldsymbol{l}_i$ is a principal direction.
- Discrete filtering:

$$f_{\text{app}}^{\boldsymbol{l}_i}(\boldsymbol{x}) = \frac{1}{h} \sum_{\boldsymbol{n}} (f * q_i)[\boldsymbol{n}] \varphi(\tfrac{\boldsymbol{x}}{h} - \tfrac{\boldsymbol{l}_i}{2} - \mathbf{L}\boldsymbol{n}).$$

- Use the error kernel for derivatives to design filters that can be used on the fly.

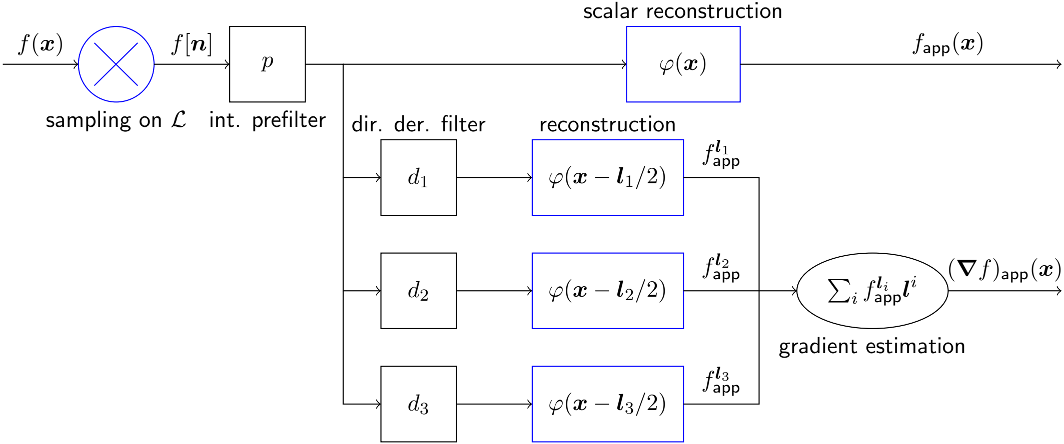
# Error Quantification

## Error Kernel for Derivatives (extension of [Condat and Möller, 2011])

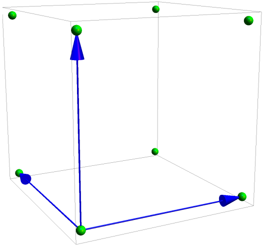
$$E^{\boldsymbol{l}_i}(\boldsymbol{\omega}) := E_{\text{min}}(\boldsymbol{\omega}) + \underbrace{\hat{A}_\varphi(\boldsymbol{\omega})\left|\frac{\hat{Q}_i(\boldsymbol{\omega})}{2\pi\imath{\boldsymbol{l}_i}^{\mathsf{T}}\boldsymbol{\omega}} - \hat{\mathring{\varphi}}(\boldsymbol{\omega})\exp(\pi\imath{\boldsymbol{l}_i}^{\mathsf{T}}\boldsymbol{\omega})\right|^2}_{E_{\text{res}}^{\boldsymbol{l}_i}(\boldsymbol{\omega})}.$$

- $q_i$ is applied to the samples of $f$.
- **Optimality criterion:** $E_{\text{res}}^{\boldsymbol{l}_i}(\boldsymbol{\omega}) = O(\|\boldsymbol{\omega}\|^{2k})$, where $k$ is the approximation order.

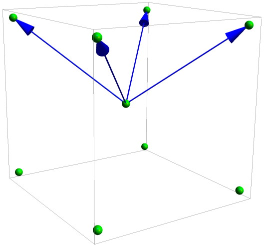
# Interpolative Model: $q_i[\boldsymbol{n}] = (p * d_i)[\boldsymbol{n}]$



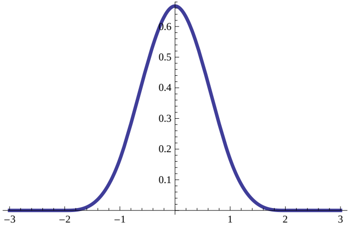
# Fourth-order FIR Filters



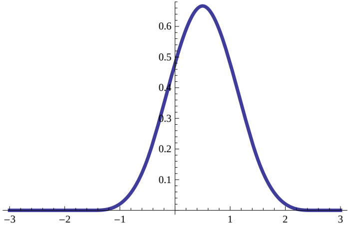
CC



BCC




Centered (*pFIR*): $[-\frac{1}{12}, \frac{2}{3}, 0, -\frac{2}{3}, \frac{1}{12}]$

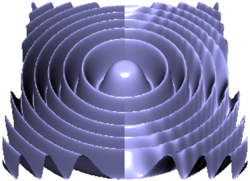



Shifted (*pFIR-s*): $[-\frac{1}{24}, \frac{9}{8}, -\frac{9}{8}, \frac{1}{24}, 0]$

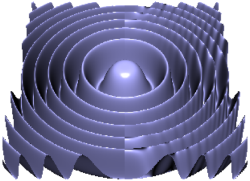
# Tricubic B-spline (CC) vs. Quintic Box Spline (BCC)

**Centered** | **Shifted**

CC

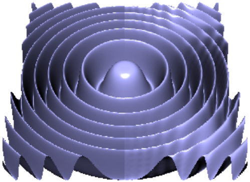


*pFIR*, $24.14°, 2.11$

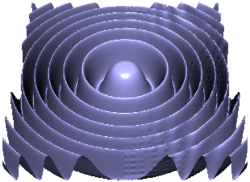


*pFIR-s*, $11.53°, 1.15$

BCC



*P-OPT26*, $18.10°, 1.96$



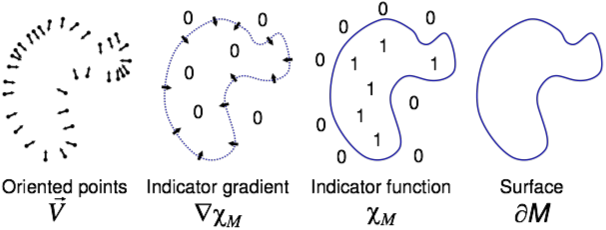
*P-FIR-s*, $8.7°, 1.03$

Mean angular and magnitude errors are indicated

# Outline

# Motivation

- Surface reconstruction as a Poisson problem.




$$\Delta\chi_M = \vec{\nabla}\cdot\vec{V}$$

# Homogeneous Poisson Equation

## Problem

$$\Delta V = f \text{ in } \mathcal{C}^s,$$
$$V = 0 \text{ on } \partial\mathcal{C}^s.$$

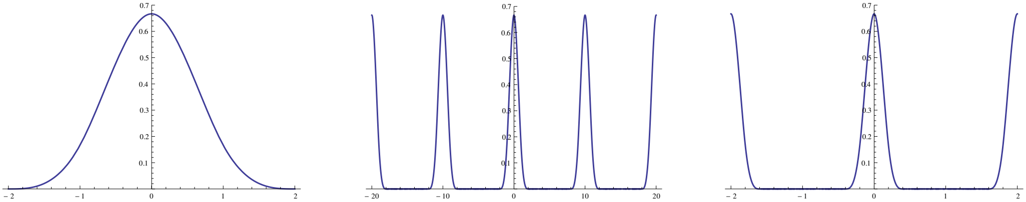
Given lattice samples of $f$ inside $\mathcal{C}^s$, approximate $V$.

## Analytic Solution

- $-\tilde{V}[\boldsymbol{m}] = \dfrac{\tilde{f}[\boldsymbol{m}]}{\pi^2\|\boldsymbol{m}\|^2}$ where $\boldsymbol{m} \in \mathbb{Z}_+^s$.
- Solution operator: $\Delta^{-1} \Leftrightarrow (\pi^2\|\boldsymbol{m}\|^2)^{-1}$.
- $V$ can be extended so that it is $\mathcal{P}^s$-periodic, where $\mathcal{P}^s := [-1, 1]^s$.

# Key Observations

- Seek an approximation $V_{\text{app}} \in \mathbb{V}(\mathcal{L}_h, \varphi_p)$.
- Periodic generator: $\varphi_p(\boldsymbol{x}) := \sum_{\boldsymbol{m}\in\mathbb{Z}^s} \varphi(\boldsymbol{x} - \frac{2}{h}\boldsymbol{m})$.
- Finite summation:
$$V_{\text{app}}(\boldsymbol{x}) = \sum_{\boldsymbol{x}_j \in \mathcal{P}_h} c[\boldsymbol{x}_j]\varphi_p(\frac{\boldsymbol{x} - \boldsymbol{x}_j}{h}).$$
- *Dirichlet boundary conditions can be imposed by requiring that $c[\cdot]$ be odd.*




(a) $\beta^3(x)$ (b) $\beta_p^3(x)$ (c) $\beta_p^3(\frac{x}{h})$

$$h = 0.2$$

# Key Observations

- Seek an approximation $V_{\text{app}} \in \mathbb{V}(\mathcal{L}_h, \varphi_p)$.
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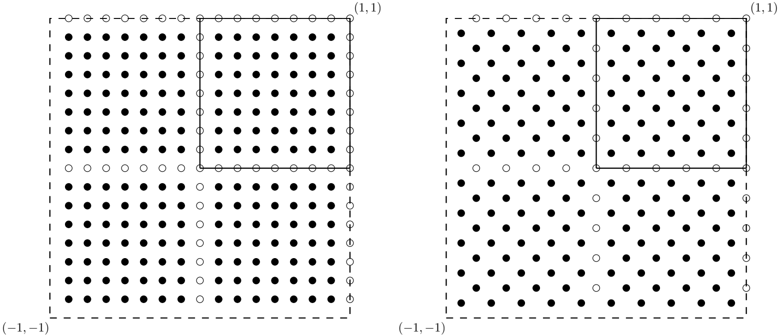



Samples inside $\mathcal{P}^s$

# Solution Methodology

- Discrete filtering: $c[\boldsymbol{x}_j] = (f \circledast q)[\boldsymbol{x}_j]$,
  and $q[\cdot] \leftrightarrow \hat{Q}(\boldsymbol{\omega})$ is a suitable discretization of $\Delta^{-1} \leftrightarrow (-4\pi^2\|\boldsymbol{\omega}\|^2)^{-1}$.

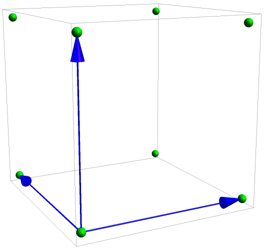
## Error Quantification (extension of [Jacob et al., 2002])

The error $\|V - V_{\mathsf{app}}\|_{L_2(\mathcal{P}^s)}$ can be predicted through

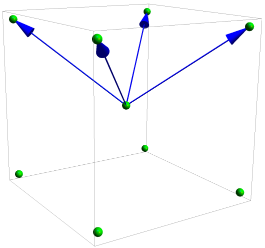
$$E(\boldsymbol{\omega}) = E_{\mathsf{min}}(\boldsymbol{\omega}) + \underbrace{\hat{A}_{\varphi}(\boldsymbol{\omega})\left|4\pi^2\|\boldsymbol{\omega}\|^2\hat{Q}(\boldsymbol{\omega}) + \hat{\mathring{\varphi}}(\boldsymbol{\omega})\right|^2}_{E_{\mathsf{mod}}(\boldsymbol{\omega})}.$$

# Interpolative Model: $\hat{Q}(\boldsymbol{\omega}) = \dfrac{\hat{P}(\boldsymbol{\omega})}{\hat{\Lambda}(\boldsymbol{\omega})}$

- Asymptotically optimal:
  $E_{\text{mod}}(\boldsymbol{\omega}) = O(\|\boldsymbol{\omega}\|^{2k})$ as long as $\hat{\Lambda}(\boldsymbol{\omega}) = -4\pi^2\|\boldsymbol{\omega}\|^2 + O(\|\boldsymbol{\omega}\|^{k+2})$.
- On CC and BCC, we can use a 1D filter along the principal directions.



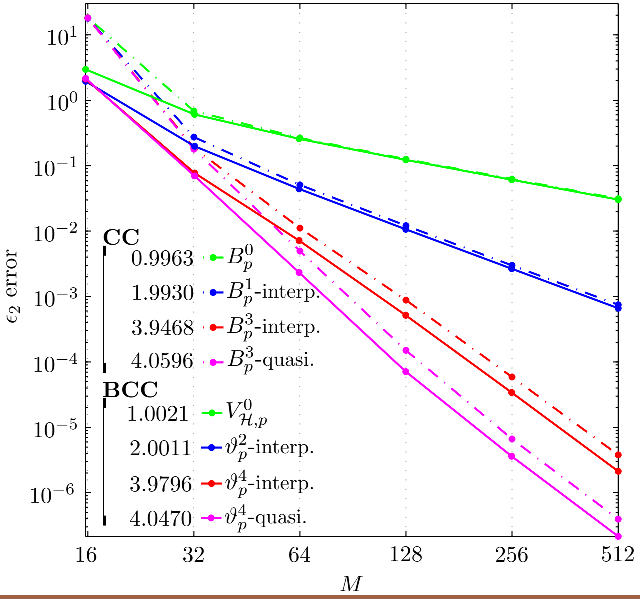
CC



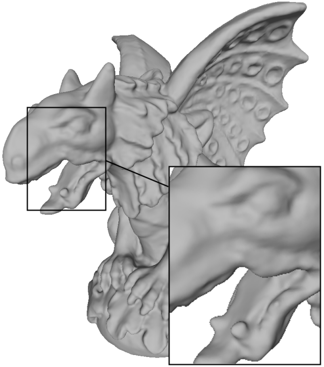
BCC

Fourth-order 1D Laplacian filter: $[-\frac{1}{12}, \frac{4}{3}, -\frac{5}{2}, \frac{4}{3}, -\frac{1}{12}]$

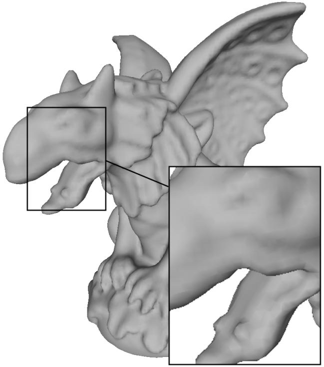
# 3D Results: $V(\boldsymbol{x}) := \sin(12\pi \sin(\pi x_1) \sin(\pi x_2) \sin(\pi x_3))$



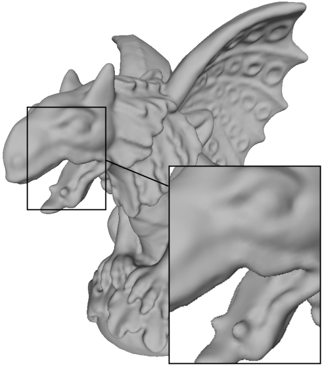
# Surface Reconstruction Results



original



[Kazhdan et al., 2006]



BCC

# Outline

# Summary

## Gradient Estimation

- Consistent gradient reconstruction — asymptotically optimal.
- Interpolative model.
- Two-stage framework — *Order of $\psi \geq$ order of $\varphi$.*
- Easy extension to other lattices and dimensions.

## Other Operators

- General error-kernel formulation for the discretization of shift-invariant operators.
- Consistent approximations that respect the order provided by $\varphi$.